Sample Complexity of Policy Gradient Finding Second-Order Stationary Points

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December 4, 2020

Abstract

The goal of policy-based reinforcement learning (RL) is to search the maximal point of its objective. However, due to the inherent non-concavity of its objective, convergence to a first-order stationary point (FOSP) can not guarantee the policy gradient methods finding a maximal point. A FOSP can be a minimal or even a saddle point, which is undesirable for RL. Fortunately, if all the saddle points are *strict*, all the second-order stationary points (SOSP) are exactly equivalent to local maxima. Instead of FOSP, we consider SOSP as the convergence criteria to character the sample complexity of policy gradient. Our result shows that policy gradient converges to an $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP with probability at least $1 - \widetilde{\mathcal{O}}(\delta)$ after the total cost of $\mathcal{O}\left(\frac{\epsilon^{-\frac{9}{2}}}{(1-\gamma)\sqrt{\chi}}\log\frac{1}{\delta}\right)$, where $\gamma \in (0, 1)$. Our result improves the state-of-the-art result

 $(1-\gamma)\sqrt{\chi} = \delta$ significantly where it requires $\mathcal{O}\left(\frac{\epsilon^{-9}\chi^{\frac{3}{2}}}{\delta}\log\frac{1}{\epsilon\chi}\right)$. Our analysis is based on the key idea that

decomposes the parameter space \mathbb{R}^p into three non-intersected regions: non-stationary point, saddle point, and local optimal region, then making a local improvement of the objective of RL in each region. This technique can be potentially generalized to extensive policy gradient methods.

1 Introduction

Policy gradient method (Williams, 1992; Sutton et al., 2000) is widely used to search the optimal policy in modern reinforcement learning (RL). Such method (or its variants) searches over a differentiable parameterized class of polices by performing a stochastic gradient on a cumulative expected reward function. Due to its merits such as the simplicity of implementation in the simulated environment; it requires low memory; it can be applied to any differentiable parameterized classes (Agarwal et al., 2019), policy gradient method has achieved significant successes in challenging fields such as robotics (Deisenroth et al., 2013; Duan et al., 2016), playing Go (Silver et al., 2016, 2017), neural architecture search (Zoph and Le, 2017), NLP (Kurita and Søgaard, 2019; Whiteson, 2019), computer vision (Sarmad et al., 2019), and recommendation system (Pan et al., 2019).

Despite it has tremendous successful applications, suffering from high sample complexity is still a critical challenge for the policy gradient (Haarnoja et al., 2018; Lee et al., 2019; Xu et al., 2020a). Thus, for policy gradient, theory analysis of its sample complexity plays an important role in RL

since the sample complexity not only provides an understanding of the policy gradient but also gives insights on how to improve the sample efficiency of the existing RL algorithms.

Investigation of the sample complexity of policy gradient algorithm (or its variant) can be traced back to the pioneer works of Kearns et al. (2000); Kakade (2003). Recently, to improve sample efficiency, Papini et al. (2018); Shen et al. (2019); Xu et al. (2020a) introduce stochastic variance reduced gradient techniques (Johnson and Zhang, 2013; Nguyen et al., 2017a) to policy optimization, and they have studied the sample complexity of policy gradient methods to achieve a first-order stationary point (FOSP) (i.e., θ such that $\|\nabla J(\theta)\|_2 \leq \epsilon$). However, since the objective of RL is a non-concave function with respect to the standard policy parameterizations (Papini et al., 2018; Agarwal et al., 2019), a FOSP could be a maximal point, a minimal point, and even a saddle point. Both minimal points and saddle points are undesirable for policy gradient since its goal is to search a maximal point, which implies within the numbers of samples provided by Papini et al. (2018); Shen et al. (2019); Xu et al. (2020a), we can not guarantee the output of their policy gradient algorithm is a maximal point. This motivates a fundamental question as follows,

Question 1. How many samples does an agent need to collect to guarantee the policy gradient methods converge to a maximal point certainly?

1.1 Our Work

In this paper, we consider the second-order stationary point (SOSP) to answer Question 1. More specifically, inspired by the previous works from non-convex optimization (Jin et al., 2017; Daneshmand et al., 2018), we investigate the sample complexity of policy gradient methods finding an $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP, see Definition 1, i.e., the convergent point θ satisfies

$$\|\nabla J(\theta)\|_2 \le \epsilon$$
, and $\lambda_{\max}(\nabla^2 J(\theta)) \le \sqrt{\chi \epsilon}$.

The criterion of $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP requires the convergent point with a small gradient and with almost a negative semi-definite Hessian matrix. This criterion not only ensures a convergent point is a FOSP but also rules out both saddle points (whose Hessian are indefinite) and minimal points (whose Hessian are positive definite). Therefore, convergence to a $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP guarantees the policy gradient methods converge to a local maximal point clearly. Our result shows that within a cost of

$$\mathcal{O}\left(\frac{\epsilon^{-\frac{9}{2}}}{(1-\gamma)\sqrt{\chi}}\log\frac{1}{\delta}\right) = \widetilde{\mathcal{O}}\left(\epsilon^{-\frac{9}{2}}\right),$$

policy gradient converges to an $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP with probability at least $1 - \tilde{\mathcal{O}}(\delta)$. Our result improves the state-of-the-art result of (Zhang et al., 2019) significantly, where they require $\tilde{\mathcal{O}}(\epsilon^{-9})$ samples to achieve an $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP.

Notably, we provide a novel analysis that can be potentially generalized to extensive policy gradient methods. Concretely, we decompose the parameter space \mathbb{R}^p into three different regions: non-stationary point, saddle point, and local optimal region, then making a local improvement in each region. The main challenge occurs on the saddle point region, where we utilize a technique called correlated negative curvature (CNC) (Daneshmand et al., 2018) to make a local improvement.

1.2 Paper Organization

In Section 2, we introduce some necessary conceptions of policy gradient and some standard assumptions in policy optimization. In Section 3, we formally define $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP. Our main contribution

lies in Section 4, where we provide the main result that presents the sample complexity of policy gradient finding an $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP, and we provide an overview of the proof technique. Related works and future works are discussed in Section 5.

1.3 Notations

Let $\|\cdot\|_2$ be the Euclidean norm of a vector in \mathbb{R}^p . For a symmetric matrix A, we use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ as its minimum and maximum eigenvalue correspondingly. Let $\|A\|_{op}$ denote the operator norm of the matrix A; furthermore, according to Van Loan and Golub (1983), if $A \in \mathbb{R}^{p \times p}$ is a symmetric matrix, then $\|A\|_{op} = \max_{1 \le i \le p} \{|\lambda_i|\}$, where $\{\lambda_i\}_{i=1}^p$ is the set of the eigenvalues of A. We use $A \succ 0$ to denote a positive definite matrix A. For a function $J(\cdot) : \mathbb{R}^p \to \mathbb{R}$, let ∇J and $\nabla^2 J$ denote its gradient vector and Hessian matrix correspondingly. Let $\mathbb{B}_2(o, r)$ be a p-dimensional ℓ_2 ball with the centre o and radius r, i.e., $\mathbb{B}_2(o, r) = \{x \in \mathbb{R}^p; \|x - o\|_2 \le r\}$. For any real number x, [x] and [x] denote the nearest integer to x from above and below. We use \widetilde{O} to hide polylogarithmic factors in the input parameters, i.e., $\widetilde{O}(f(x)) = O(f(x)\log(f(x))^{O(1)})$.

2 Policy Gradient Methods and Some Standard Assumptions

In this section, we introduce some necessary concepts of reinforcement learning, policy gradient and some standard assumptions in policy optimization.

2.1 Reinforcement Learning

Reinforcement learning (Sutton and Barto, 2018) is often formulated as Markov decision processes (MDP) $\mathcal{M} = (\mathcal{S}, \mathcal{A}, P, R, \rho_0, \gamma)$, where \mathcal{S} is the state space, \mathcal{A} is the action space; P(s'|s, a) is the probability of state transition from s to s' under playing the action a; $R(\cdot, \cdot) : \mathcal{S} \times \mathcal{A} \to [R_{\min}, R_{\max}]$ is a bounded reward function, where R_{\min}, R_{\max} two positive scalars. $\rho_0(\cdot) : \mathcal{S} \to [0, 1]$ is the initial state distribution and the discount factor $\gamma \in (0, 1)$.

The parametric policy π_{θ} is a probability distribution over $S \times A$ with a parameter $\theta \in \mathbb{R}^p$, and we use $\pi_{\theta}(a|s)$ to denote the probability of playing a in state s. Let $\tau = \{s_t, a_t, r_{t+1}\}_{t\geq 0} \sim \pi_{\theta}$ be a trajectory generated by the policy π_{θ} , where $s_0 \sim \rho_0(\cdot)$, $a_t \sim \pi_{\theta}(\cdot|s_t)$, $r_{t+1} = R(s_t, a_t)$ and $s_{t+1} \sim P(\cdot|s_t, a_t)$. The state value function of π_{θ} is defined as follows,

$$V^{\pi_{\theta}}(s) = \mathbb{E}_{\pi_{\theta}}\left[\sum_{t=0}^{\infty} \gamma^{t} r_{t+1} | s_{0} = s\right],$$

where $\mathbb{E}_{\pi_{\theta}}[\cdot|\cdot]$ denotes a conditional expectation on actions which are selected according to the policy π_{θ} . The *advantage function* of the policy π_{θ} is defined as follows,

$$A^{\pi_{\theta}}(s,a) = Q^{\pi_{\theta}}(s,a) - V^{\pi_{\theta}}(s),$$

where $Q^{\pi_{\theta}}(s, a)$ is the state-action value function:

$$Q^{\pi_{\theta}}(s,a) = \mathbb{E}_{\pi_{\theta}}\left[\sum_{t=0}^{\infty} \gamma^{t} r_{t+1} | s_{0} = s, a_{0} = a\right].$$

We use $P^{\pi_{\theta}}(s_t = s | s_0)$ to denote the probability of visiting the state s after t time steps from the initial state s_0 by executing π_{θ} , and

$$d_{s_0}^{\pi_\theta}(s) = \sum_{t=0}^{\infty} \gamma^t P^{\pi_\theta}(s_t = s|s_0)$$

is the (unnormalized) discounted stationary state distribution of the Markov chain (starting at s_0) induced by π_{θ} . Furthermore, since $s_0 \sim \rho_0(\cdot)$, we define

$$d_{\rho_0}^{\pi_\theta}(s) = \mathbb{E}_{s_0 \sim \rho_0(\cdot)}[d_{s_0}^{\pi_\theta}(s)]$$

as the discounted state visitation distribution over the initial distribution ρ_0 . Recall $\tau = \{s_t, a_t, r_{t+1}\}_{t \ge 0} \sim \pi_{\theta}$, we define

$$J(\pi_{\theta}|s_0) = \mathbb{E}_{\tau \sim \pi_{\theta}, s_0 \sim \rho_0(\cdot)}[R(\tau)] = \mathbb{E}_{s \sim d_{s_0}^{\pi_{\theta}}(\cdot), a \sim \pi_{\theta}(\cdot|s)}[R(s, a)],$$

where $R(\tau) = \sum_{t\geq 0} \gamma^t r_{t+1}$, and $J(\pi_{\theta}|s_0)$ is "conditional" on s_0 since we emphasize the trajectory τ starting from s_0 . Furthermore, we define the expected return $J(\theta) =: \mathbb{E}_{s_0 \sim \rho_0(\cdot)}[J(\pi_{\theta}|s_0)]$ as follows,

$$J(\theta) = \mathbb{E}_{s \sim d_{\rho_0}^{\pi_{\theta}}(\cdot), a \sim \pi_{\theta}(\cdot|s)}[R(s,a)] = \int_{s \in \mathcal{S}} d_{\rho_0}^{\pi_{\theta}}(s) \int_{a \in \mathcal{A}} \pi_{\theta}(a|s)R(s,a) \mathrm{d}a\mathrm{d}s.$$
(1)

The goal of policy-based reinforcement learning is to solve the following policy optimization problem:

$$\max_{\theta \in \mathbb{R}^p} J(\theta). \tag{2}$$

2.2 Policy Gradient Methods

The basic idea of policy gradient (Williams, 1992; Sutton et al., 2000) is to update the parameter according to the direction with respect to the gradient of $J(\theta)$, i.e.,

$$\theta_{k+1} = \theta_k + \alpha \widehat{\nabla J(\theta_k)},\tag{3}$$

where $\alpha > 0$ is step-size, $\nabla J(\theta_k)$ is a stochastic estimator of policy gradient $\nabla J(\theta_k)$. According to Sutton et al. (2000), we present the well-known *policy gradient theorem* as follows,

$$\nabla J(\theta) = \int_{s \in \mathcal{S}} d_{\rho_0}^{\pi_{\theta}}(s) \int_{a \in \mathcal{A}} Q^{\pi_{\theta}}(s, a) \nabla \pi_{\theta}(s, a) \mathrm{d}a \mathrm{d}s = \mathbb{E}_{s \sim d_{\rho_0}^{\pi_{\theta}}(\cdot), a \sim \pi_{\theta}(\cdot|s)} \left[Q^{\pi_{\theta}}(s, a) \nabla \log \pi_{\theta}(a|s) \right],$$

which provides a possible way to find the estimator of $\nabla J(\theta)$. One issue that we should address is how to estimate $Q^{\pi_{\theta}}(s, a)$ appears in the policy gradient theorem. A simple approach is to use a sample return $R(\tau)$ to estimate $Q^{\pi_{\theta}}(s, a)$, i.e., we calculate the policy gradient estimator as follows,

$$g(\tau|\theta) = \sum_{t \ge 0} \nabla \log \pi_{\theta}(a_t|s_t) R(\tau).$$
(4)

Replace $\nabla \widehat{J(\tau|\theta_k)}$ of (3) with $g(\tau|\theta_k)$, we achieve the update rule of REINFORCE (Williams, 1992):

$$\theta_{k+1} = \theta_k + \alpha g(\tau | \theta_k). \tag{5}$$

2.3 Fisher Information Matrix

For the policy optimization (1), we learn the parameter from the samples that come from an unknown probability distribution. Fisher information matrix (Fisher, 1920; Kakade, 2002; Ly et al., 2017) provides the information that a sample of data provides about the unknown parameter. According to Kakade (2002); Bhatnagar et al. (2008), the Fisher information matrix $F(\theta)$ is positive definite, i.e., there exists a constant $\omega > 0$ s.t.,

$$F(\theta) =: \int_{s \in \mathcal{S}} d_{\rho_0}^{\pi_{\theta}}(s) \int_{a \in \mathcal{A}} \nabla \log \pi_{\theta}(a|s) [\nabla \log \pi_{\theta}(a|s)]^{\top} \mathrm{d}s \mathrm{d}a \succ \omega I_p, \quad \forall \ \theta \in \mathbb{R}^p, \tag{6}$$

where $I_p \in \mathbb{R}^{p \times p}$ is the identity matrix.

2.4 Standard Assumptions

Assumption 1. For each pair $(s, a) \in S \times A$, for any $\theta \in \mathbb{R}^p$, and all components *i*, *j*, there exists positive two constants $0 \leq G, L, U < \infty$ such that

(a) :
$$\left|\nabla_{\theta_i} \log \pi_{\theta}(a|s)\right| \le G$$
; (b) : $\left|\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log \pi_{\theta}(a|s)\right| \le L$; (c) : $\left|\nabla_{\theta_i} \pi_{\theta}(a|s)\right| \le U$. (7)

Assumptions 1 is a standard condition in policy optimization, and it has be applied to several recent policy gradient literatures (Castro and Meir, 2010; Pirotta et al., 2015; Papini et al., 2018; Shen et al., 2019; Xu et al., 2020a). Assumption 1 is reasonable a condition since the widely used policy classes such as Gaussian, softmax (Konda and Borkar, 1999), and relative entropy policy (Peters et al., 2010) all satisfy (7). Recently, Zhang et al. (2019); Papini et al. (2019); Wang and Zou (2020) have provided the details to check above policies satisfy Assumptions 1.

According to the Lemma B.2 of (Papini et al., 2018), Assumption 1 implies the expected return $J(\theta)$ is ℓ -Lipschitz smooth, i.e., for any $\theta, \theta' \in \mathbb{R}^p$, we have

$$\|\nabla J(\theta) - \nabla J(\theta')\|_2 \le \ell \|\theta - \theta'\|_2,\tag{8}$$

where $\ell = \frac{R_{\max}h(hG^2+L)}{(1-\gamma)}$, *h* is a positive scalar that denotes the horizon of the trajectory τ . The property (8) has been given as the Lipschitz assumption in previous works (Kumar et al., 2019; Wang et al., 2020), and it has been also verified by lots of recent works with some other regularity conditions (Zhang et al., 2019; Agarwal et al., 2019; Xu et al., 2020b).

Furthermore, according to the Lemma 4.1 of (Shen et al., 2019), Assumption 1 implies a property of the policy gradient estimator as follows, for each $\tau \sim \pi_{\theta}$, we have

$$\|g(\tau|\theta) - \nabla J(\theta)\|_2 \le \frac{GR_{\max}}{(1-\gamma)^2} =: \sigma.$$
(9)

The result of (9) implies the boundedness of the variance of the policy gradient estimator $g(\tau|\theta)$, i.e., $\operatorname{Var}(g(\tau|\theta)) = \mathbb{E}[\|g(\tau|\theta) - \nabla J(\theta)\|_2^2] \leq \sigma^2$. The boundedness of $\operatorname{Var}(g(\tau|\theta))$ are also proposed as an assumption in the previous works (Papini et al., 2018; Xu et al., 2019, 2020a; Wang et al., 2020).

Assumption 2 (Smoothness of Policy Hessian). The the expected return function $J(\theta)$ is χ -Hessian-Lipschitz, i.e., there exists a constant $0 \leq \chi < \infty$ such that for all $\theta, \theta' \in \mathbb{R}^p$:

$$\|\nabla^2 J(\theta) - \nabla^2 J(\theta')\|_{op} \le \chi \|\theta - \theta'\|_2.$$
⁽¹⁰⁾

Assumption 2 requires that for the two near points, the Hessian matrix $\nabla^2 J(\cdot)$ can not change dramatically in the terms of operator norm. For RL, the parameter χ can be deduced by some other regularity conditions, e.g., Zhang et al. (2019) provides an estimation of χ , see Appendix B.

3 Second-Order Stationary Point

Due to the non-concavity of $J(\theta)$, finding global maxima is NP-hard in the worst case. The best one can hope is to convergence to stationary points. In this section, we formally define second-order stationary point (SOSP). Furthermore, with the second-order information, we present Assumption 3 to make clear the maximal point that we mainly concern for policy optimization. **Definition 1** (Second-Order Stationary Point (Nesterov and Polyak, 2006)¹). For the χ -Hessian-Lipschitz function $J(\cdot)$, we say that θ is a second-order stationary point if

 $\|\nabla J(\theta)\|_2 = 0 \quad \text{and} \quad \lambda_{\max}(\nabla^2 J(\theta)) \le 0; \tag{11}$

we say θ is an $(\epsilon, \sqrt{\chi \epsilon})$ -second-order stationary point if

$$\|\nabla J(\theta)\|_2 \le \epsilon \quad \text{and} \quad \lambda_{\max}(\nabla^2 J(\theta)) \le \sqrt{\chi \epsilon}.$$
 (12)

The SOSP is a very important concept for the policy optimization (2) because it rules the saddle points (whose Hessian are indefinite) and minimal points (whose Hessian are positive definite), which is usually more desirable than convergence to a first-order stationary point (FOSP). Recently, Shen et al. (2019); Xu et al. (2020a) introduce FOSP to measure the convergence of policy gradient methods. As mentioned in Section 1, for policy optimization (2), an algorithm converges to a FOSP is not sufficient to ensure that algorithm outputs a maximal point. While SOSP overcomes above shortcomings, which is our main motivation to consider SOSP as a convergence criterion.

Assumption 3 (Structure of $J(\theta)$). For any $\theta \in \mathbb{R}^p$, at least one of the following holds: (i) $\|\nabla J(\theta)\| \geq \epsilon$; (ii) $\lambda_{\max}(\nabla^2 J(\theta)) \geq \sqrt{\epsilon \chi}$; (iii) θ nears a local maximal point θ_{\star} : there exists a positive scalar ρ such that θ falls in to the ball $\mathbb{B}_2(\theta_{\star}, \rho)$, and $J(\theta)$ is ζ -strongly concave on $\mathbb{B}_2(\theta_{\star}, \rho)$.

In the standard non-convex literature such as (Ge et al., 2015; Jin et al., 2017), the condition (ii) of Assumption 3 is often called $(\epsilon, \chi, \varrho)$ -strict saddle. In this case, all the SOSP are local maxima and hence convergence to second-order stationary points is equivalent to convergence to local maxima. In the following three-states MDP (see Figure 1), we verify that the Assumption 3 holds on policy optimization.



Figure 1: Three-States MDP.

Example 1. In this deterministic MDP, the states s_1 and s_2 equip the action space $\mathcal{A} = \{ \texttt{right}, \texttt{left} \}$, while s_0 equips an additional action up. The policy $\pi_{\theta}(\cdot|\cdot)$ with a parameter $\theta = (\theta_1, \theta_2)^{\top} \in \mathbb{R}^2$, Let $C_o =: [0,1] \times [0,1]$, if $\theta \in C_o$, we define $\pi_{\theta}(\texttt{right}|s_0) =: p_1 = \frac{1}{\sqrt{2\pi}}(1-\theta_1^2+\theta_2^2)$; if $\theta \notin C_o$, we define Gaussian policy $\pi_{\theta}(\texttt{left}|s_0) =: p_2 = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(2-\|\theta\|_2^2)}{2}\right\}$; otherwise, $\pi_{\theta}(\texttt{up}|s_0) =: p_3 = 1-p_1-p_2$. Then $J(\theta) = p_1R_1 + p_2R_2 + p_3R_3$, i.e.,

$$J(\theta) = \begin{cases} \frac{1}{\sqrt{2\pi}} (1 - \theta_1^2 + \theta_2^2), & \theta \in C_o \\ \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(2 - \|\theta\|_2^2)}{2}\right\}, & \theta \notin C_o. \end{cases}$$
(13)

The function $J(\theta)$ (13) satisfies Assumption 3. Since the origin $(0,0) \in C_o$ is a saddle point of $J(\theta)$, and $\lambda_{\max}(\nabla^2 J(\theta)|_{(0,0)}) = 1$, thus the point (0,0) satisfies strict saddle point property. Besides, on the complementary space of C_o , i.e., $\mathbb{R}^2 - C_o$, $J(\theta)$ is a strongly concave function.

¹Recall problem (2) is a maximization problem, thus this definition of SOSP is slightly different from the minimization problem $\min_x f(x)$, where it requires $\|\nabla f(x)\|_2 \leq \epsilon$ and $\lambda_{\min}(\nabla^2 f(x)) \geq 0$. Similarly, its $(\epsilon, \sqrt{\chi\epsilon})$ -SOSP requires $\|\nabla f(x)\|_2 \leq \epsilon$ and $\lambda_{\min}(\nabla^2 f(x)) \geq -\sqrt{\chi\epsilon}$.

4 Main Result and Technique Overview

Our contribution lies in this section. Theorem 1 presents the sample complexity of policy gradient algorithm (5) finding an $(\epsilon, \sqrt{\chi\epsilon})$ -SOSP. Section 4.1 provides an overview of the proof technique; Section 4.1.1 provides all the key steps, Section 4.1.2 provides a sketch of the proof of Theorem 1.

Theorem 1. Under Assumption 1-3, consider $\{\theta_k\}_{k\geq 0}$ generated by (5), and ι is defined in (20). For a small enough step-size α such that $\alpha \leq \min\left\{\frac{\epsilon^2}{2\sqrt{\chi\epsilon}R_{\min}^2\omega^2}, \frac{2\epsilon^2}{(\epsilon^2+\sigma^2)\ell}\right\} = \mathcal{O}(\epsilon^2)$, the iteration (5) returns an $(\epsilon, \sqrt{\chi\epsilon})$ -SOSP with probability at least $1 - \delta - \delta \log(\frac{1}{\delta}) = 1 - \widetilde{O}(\delta)$ after the times of

$$K = \left\lceil \frac{6R_{\max}}{\alpha^2(1-\gamma)\iota^2\sqrt{\chi\epsilon}}\log\frac{1}{\delta} \right\rceil + 1 = \mathcal{O}\left(\frac{\epsilon^{-\frac{9}{2}}}{(1-\gamma)\sqrt{\chi}}\log\frac{1}{\delta}\right).$$

Remark 1. Theorem 1 illustrates that policy gradient algorithm (5) needs a cost of $\widetilde{\mathcal{O}}(\epsilon^{-\frac{9}{2}})$ to find an $(\epsilon, \sqrt{\chi\epsilon})$ -SOSP. To the best of our knowledge, Zhang et al. (2019) firstly consider to introduce SOSP to measure the sample complexity of policy-based RL. Zhang et al. (2019) propose a *modified* random-horizon policy gradient (MRPG) algorithm, and they show that MRPG needs at least a cost of $\mathcal{O}(\epsilon^{-9}\chi^{\frac{3}{2}}\frac{1}{\delta}\log\frac{1}{\epsilon\chi}) = \widetilde{\mathcal{O}}(\epsilon^{-9})$ to find an $(\epsilon, \sqrt{\chi\epsilon})$ -SOSP. Clearly, result of Theorem 1 improves the sample complexity of Zhang et al. (2019) significantly from $\widetilde{\mathcal{O}}(\epsilon^{-9})$ to $\widetilde{\mathcal{O}}(\epsilon^{-\frac{9}{2}})$. Additionally, compared to Zhang et al. (2019), our analysis does *not* invoke a geometric distribution restriction on the horizon. In the real work, the horizon of a trajectory only depends on the simulated environment, it is not necessary to draw a horizon from a geometric distribution, i.e., our result is more practical.

4.1 Technique Overview

Recall that an $(\epsilon, \sqrt{\chi \epsilon})$ -SOSP requires a point has a small gradient, and whose Hessian matrix does not have a significantly positive eigenvalue, which inspires us to consider an idea that decomposes the parameter space \mathbb{R}^p into three non-intersected regions, and then analyzing them separately.

O Case I: Non-Stationary Region. In this case, we consider the region with large gradient, i.e.,

$$\mathcal{L}_1 = \{ \theta \in \mathbb{R}^p : \|\nabla J(\theta)\|_2 \ge \epsilon \};$$
(14)

2 Case II: Around Saddle Point. We consider the region where the norm of the policy gradient is small, while the maximum eigenvalue of the Hessian matrix $\nabla^2 J(\theta)$ is larger than zero:

$$\mathcal{L}_2 = \{ \theta \in \mathbb{R}^p : \|\nabla J(\theta)\|_2 \le \epsilon \} \cap \{ \theta \in \mathbb{R}^p : \lambda_{\max}(\nabla^2 J(\theta)) \ge \sqrt{\chi \epsilon} \};$$
(15)

3 Case III: Local Optimal Region. In this case, we consider the region $\mathcal{L}_3 = \mathbb{R}^p - (\mathcal{L}_1 \cup \mathcal{L}_2)$:

$$\mathcal{L}_3 = \{ \theta \in \mathbb{R}^p : \|\nabla J(\theta)\|_2 \le \epsilon \} \cap \{ \theta \in \mathbb{R}^p : \lambda_{\max}(\nabla^2 J(\theta)) \le \sqrt{\chi \epsilon} \}.$$
(16)

It is noteworthy that the local optimal region, i.e., \mathcal{L}_3 is the desirable region where we expect policy gradient algorithm converges to it with high probability. Before we provide the formal proof, in Section 4.1.1, we present three separate propositions to make local improvement on above three regions correspondingly. The main challenge occurs on region \mathcal{L}_2 , where we utilize a technique called correlated negative curvature (CNC) (Daneshmand et al., 2018) to make a local improvement.

4.1.1 Local Improvement on Each Case

Proposition 1 (Local Improvement on \mathcal{L}_1). Under Assumption 1-2. The sequence $\{\theta_k\}_{k\geq 0}$ generated according to (5). If a point $\theta_k \in \mathcal{L}_1$, let $\alpha < \min\left\{\frac{2\epsilon^2}{(\epsilon^2 + \sigma^2)\ell}, \frac{2}{\ell}\right\} = \mathcal{O}(\epsilon^{-2})$, then we have

$$\mathbb{E}\left[J(\theta_{k+1})\right] - J(\theta_k) \stackrel{(\mathbf{a})}{\geq} \left(\alpha - \frac{\ell\alpha^2}{2}\right) \|\nabla J(\theta_k)\|_2^2 - \frac{\ell\alpha^2\sigma^2}{2} \stackrel{(\mathbf{b})}{\geq} \frac{1}{2}\alpha\epsilon^2.$$
(17)

We provide its proof in Appendix C. Proposition 1 shows that when the gradient is large, the expected return $J(\theta)$ increases in one step. It is noteworthy that the step-size plays an important role in achieving the result of (17). Concretely, for a positive scalar $\alpha - \ell \alpha^2/2$ (i.e., which requires $\alpha < 2/\ell$), Eq.(a) of (17) guarantees the desired increase whenever the norm of the policy gradient is large enough. At the same time, when considering the lower threshold value ϵ of the norm of the policy gradient in the region \mathcal{L}_1 , the second term of (17) achieves at least $\alpha \epsilon^2 - \ell \alpha^2 (\epsilon^2 + \sigma^2)/2$. Thus, to make a clear improvement, the condition (b) requires step-size α should satisfy $\alpha < 2\epsilon^2/(\epsilon^2 + \sigma^2)\ell$.

Proposition 2 (Local Improvement on \mathcal{L}_2). Under Assumption 1-2, consider the sequence $\{\theta_k\}_{k\geq 0}$ generated by (5). If a point θ_k falls in to \mathcal{L}_2 , there exists a positive scalar ι (20), and $\hat{\kappa}_0$ such that

$$\widehat{\kappa}_0 =: \left\lfloor \frac{\log\left(1/(1 - \sqrt{\alpha}\sigma_{H_0})\right)}{\log(1 + \alpha\sqrt{\chi\epsilon})} \right\rfloor = \mathcal{O}(\epsilon^{-\frac{1}{2}}), \tag{18}$$

where $\sigma_{H_0} = \frac{2p\sqrt{p}hR_{\max}(hG^2+L)}{1-\gamma}$, then after at most $j \leq \hat{\kappa}_0$ steps, we have

$$\mathbb{E}[J(\theta_{k+j})] - J(\theta_k) \ge \alpha^2 \iota^2 \sqrt{\chi \epsilon}.$$
(19)

Proposition 2 illustrates that even a point gets stuck in the region that nears a saddle point, policy gradient method will ensure an increase in the value of $J(\theta)$ within at most $\mathcal{O}(\epsilon^{-\frac{1}{2}})$ steps. We provide proof of Proposition 2 in Appendix E. The proof is very technical, the following correlated negative curvature (CNC) condition (Daneshmand et al., 2018) plays a crucial role in achieving the result of (19). Concretely, let u_p be the unit eigenvector corresponding to the maximum eigenvalue of $\nabla^2 J(\theta)$, CNC ensures the second moment of the projection of policy gradient estimator $g(\tau|\theta)$ along the direction u_p is uniformly bounded away from zero, i.e., there exists a positive scalar ι s.t.,

CNC:
$$\mathbb{E}[\langle g(\tau|\theta), u_p \rangle^2] \ge \iota^2, \quad \forall \theta \in \mathbb{R}^p.$$
 (20)

We provide (20) in the Discussion 1 of Appendix E.8. CNC shows that the perturbation caused by a stochastic policy gradient estimator $g(\tau|\theta)$ is guaranteed to take an increase in the value $J(\theta)$.

Proposition 3. Under Assumption 1-3, consider the sequence $\{\theta_k\}_{k\geq 0}$ generated by (5). For any $\delta \in (0,1)$, θ_{\star} satisfies Assumption 3, let the step-size α and the stopping time κ_0 statisfy

$$\alpha \le \min\left\{\delta, \frac{1}{\zeta}, \frac{\zeta}{\ell^2}, \frac{\zeta\varrho^2}{3\sigma^2}\right\}, \alpha \log\frac{1}{\alpha} \le \frac{2\zeta\varrho^4}{27\left(G^2 R_{\max}^2/(1-\gamma)^2 + \zeta\varrho^2 + \sigma^2\right)^2}, \kappa_0 = \left\lfloor\frac{1}{\alpha^2}\log\frac{1}{\delta}\right\rfloor$$

and if some iteration θ_k falls into the ball $\mathbb{B}_2(\theta_\star, \frac{\sqrt{3}}{3}\varrho) \subset \mathbb{B}_2(\theta_\star, \varrho)$, i.e., $\|\theta_k - \theta_\star\|_2^2 \leq \frac{1}{3}\varrho^2$. Then,

$$\mathbb{P}(\left\|\theta_{k+j} - \theta_{\star}\right\|_{2} \le \varrho) \ge 1 - \delta \log \frac{1}{\delta}, \ \forall \ j \in [0, \kappa_{0} - 1].$$

We provide its proof in Appendix D. Proposition 3 illustrates that once an iteration gets sufficiently close to a local optimum θ_{\star} , it can get trapped in the neighborhood of θ_{\star} for a really long time.

4.1.2 Proof Sketch of Theorem 1

Proof. Our proof contains three steps.

- 1. Firstly, we will prove that within $\left[\frac{6R_{\max}}{\alpha^2(1-\gamma)\iota^2\sqrt{\chi\epsilon}}\right]$ steps, with probability at least $\frac{1}{2}$, one iteration falls into \mathcal{L}_3 . Let above procedure lasts $\left[\log\frac{1}{\delta}\right]$ steps, according to the *inclusion-exclusion* formula of probability: after $K_o = \left[\frac{6R_{\max}}{\alpha^2(1-\gamma)\iota^2\sqrt{\chi\epsilon}}\log\frac{1}{\delta}\right]$ steps, with probability $1-\delta$, one of $\{\theta_k\}_{k\geq 0}$ falls into \mathcal{L}_3 .
- 2. Secondly, Proposition 3 shows that once an iteration enters the region \mathcal{L}_3 , the iteration gets trapped there for at least κ_0 steps with probability $1 \delta \log \frac{1}{\delta}$.
- 3. Finally, let $K \in (K_o + 1, K_o + \kappa_0)$, combining above two results, the output θ_K falls into the region \mathcal{L}_3 with probability at least $1 (\delta + \delta \log \frac{1}{\delta})$, which concludes the result of Theorem 1.

Now, we only need to prove: starting from any point, within $\left\lceil \frac{6R_{\max}}{\alpha^2(1-\gamma)\iota^2\sqrt{\chi\epsilon}} \right\rceil$ steps, with probability at least $\frac{1}{2}$, one of $\{\theta_k\}_{k\geq 0}$ falls into \mathcal{L}_3 .

We define a stochastic process $\{\varsigma_k\}_{k\geq 0}$ ($\varsigma_0 = 0$) to trace the numbers of samples,

$$\varsigma_{k+1} = \begin{cases} \varsigma_k + 1 & \text{if } \theta_{\varsigma_k} \in \mathcal{L}_1 \cup \mathcal{L}_3\\ \varsigma_k + \widehat{\kappa}_0 & \text{if } \theta_{\varsigma_k} \in \mathcal{L}_2, \end{cases}$$

where $\hat{\kappa}_0$ is defined in (18). Let $\beta = \iota^2 \sqrt{\chi \epsilon}$, we can rewrite the results of Proposition 1-2 as follows,

$$\mathbb{E}[J(\theta_{\varsigma_{k+1}}) - J(\theta_{\varsigma_k})|\theta_{\varsigma_k} \in \mathcal{L}_1] \stackrel{(17)}{\geq} \frac{1}{2}\alpha\epsilon^2, \quad \mathbb{E}[J(\theta_{\varsigma_{k+1}}) - J(\theta_{\varsigma_k})|\theta_{\varsigma_k} \in \mathcal{L}_2] \stackrel{(19)}{\geq} \alpha^2\beta.$$
(21)

Putting the results of (21) together, let $\alpha^2\beta \leq \frac{1}{2}\alpha\epsilon^2$, i.e., $\alpha \leq \frac{\epsilon^2}{2\beta}$, we have

$$\mathbb{E}[J(\theta_{\varsigma_{k+1}}) - J(\theta_{\varsigma_k})|\theta_{\varsigma_k} \notin \mathcal{L}_3] \ge \alpha^2 \beta \mathbb{E}[(\varsigma_{k+1} - \varsigma_k)|\theta_{\varsigma_k} \notin \mathcal{L}_3].$$
(22)

We define the event \mathcal{E}_k

$$\mathcal{E}_k = \bigcap_{j=0}^k \left\{ j : \theta_{\varsigma_j} \notin \mathcal{L}_3 \right\}.$$

Let $\mathbf{1}_A$ denote indicator function, where if event A happens, $\mathbf{1}_A = 1$, otherwise $\mathbf{1}_A = 0$, then

$$\mathbb{E}[J(\theta_{\varsigma_{k+1}})\mathbf{1}_{\mathcal{E}_{k+1}} - J(\theta_{\varsigma_{k}})\mathbf{1}_{\mathcal{E}_{k}}] = \mathbb{E}[J(\theta_{\varsigma_{k+1}})(\mathbf{1}_{\mathcal{E}_{k+1}} - \mathbf{1}_{\mathcal{E}_{k}})] + \mathbb{E}[(J(\theta_{\varsigma_{k+1}}) - J(\theta_{\varsigma_{k}}))\mathbf{1}_{\mathcal{E}_{k}}] \\
\stackrel{(21)}{\geq} - \frac{R_{\max}}{1 - \gamma}(\mathbb{P}(\mathcal{E}_{k+1} - \mathcal{E}_{k})) + \alpha^{2}\beta\mathbb{E}[\varsigma_{k+1} - \varsigma_{k}|\mathbf{1}_{\mathcal{E}_{k}}]\mathbb{P}(\mathcal{E}_{k}), \quad (23)$$

where we use the boundedness of $J(\theta) \ge -\frac{R_{\max}}{1-\gamma}$.

Summing the above expectation (23) over k, then

$$\mathbb{E}[J(\theta_{\varsigma_{k+1}})\mathbf{1}_{\mathcal{E}_{k+1}}] - J(\theta_0) = -\frac{R_{\max}}{1-\gamma} (\mathbb{P}(\mathcal{E}_{k+1}) - \mathbb{P}(\mathcal{E}_0)) + \alpha^2 \beta \sum_{j=0}^k \left(\mathbb{E}[\varsigma_{j+1}]\mathbb{P}(\mathcal{E}_j) - \mathbb{E}[\varsigma_j]\mathbb{P}(\mathcal{E}_j) \right)$$

$$\geq -\frac{R_{\max}}{1-\gamma} + \alpha^2 \beta \sum_{j=0}^{k} \left(\mathbb{E}[\varsigma_{j+1}] \mathbb{P}(\mathcal{E}_{j+1}) - \mathbb{E}[\varsigma_j] \mathbb{P}(\mathcal{E}_j) \right)$$
(24)

$$= -\frac{R_{\max}}{1-\gamma} + \alpha^2 \beta \mathbb{E}[\varsigma_{k+1}] \mathbb{P}(\mathcal{E}_{k+1}), \qquad (25)$$

where Eq.(24) holds since $\mathbb{P}(\mathcal{E}_{k+1}) - \mathbb{P}(\mathcal{E}_0) \leq 1$; $\mathcal{E}_{j+1} \subset \mathcal{E}_j$ implies $\mathbb{P}(\mathcal{E}_{j+1}) \leq \mathbb{P}(\mathcal{E}_j)$; and Eq.(25) holds since $\varsigma_0 = 0$.

Finally, since $\mathbb{E}[J(\theta_{\varsigma_{k+1}})\mathbf{1}_{\mathcal{E}_{k+1}} - J(\theta_{\varsigma_k})\mathbf{1}_{\mathcal{E}_k}] \leq \frac{2R_{\max}}{1-\gamma}$, from the results of (25), if

$$\mathbb{E}[\varsigma_{k+1}] \ge \frac{6R_{\max}}{\alpha^2(1-\gamma)\beta} = \frac{6R_{\max}}{\alpha^2(1-\gamma)\iota^2\sqrt{\chi\epsilon}},$$

then $\mathbb{P}[\mathcal{E}_{k+1}] \leq \frac{1}{2}$. This concludes the proof.

5 Related Work and Future Work

Compared to the tremendous empirical works, theoretical results of policy gradient methods are relatively scarce. In this section, we compare our result with current works in the following discussion. For clarity, we have presented the complexity comparison to some results in Table 1. Furthermore, we discuss future works to extend our proof technique to other policy gradient methods.

5.1 First-Order Measurement

According to Shen et al. (2019), REINFORCE needs $\mathcal{O}(\epsilon^{-4})$ random trajectories to achieve the ϵ -FOSP, and no provable improvement on its complexity has been made so far. Xu et al. (2019) also notice the order of sample complexity of REINFORCE and GPOMDP (Baxter and Bartlett, 2001) reaches $\mathcal{O}(\epsilon^{-4})$. With an additional assumption $\mathbb{Var}\left[\prod_{i\geq 0} \frac{\pi_{\theta_0}(a_i|s_i)}{\pi_{\theta_t}(a_i|s_i)}\right], \mathbb{Var}[g(\tau|\theta)] < +\infty$, Papini et al. (2018) show that the SVRPG needs sample complexity of $\mathcal{O}(\epsilon^{-4})$ to achieve the ϵ -FOSP. Later, under the same assumption as (Papini et al., 2018), Xu et al. (2019) reduce the sample complexity of SVRPG to $\mathcal{O}(\epsilon^{-\frac{10}{3}})$. Recently, Shen et al. (2019), Yang et al. (2019a) and Xu et al. (2020a) introduce stochastic variance reduced gradient (SVRG) techniques (Johnson and Zhang, 2013; Nguyen et al., 2017a; Fang et al., 2018) to policy optimization, their new methods improve sample complexity to $\mathcal{O}(\epsilon^{-3})$ to achieve an ϵ -FOSP. Pham et al. (2020) propose ProxHSPGA that is a hybrid stochastic policy gradient estimator by combining existing REINFORCE estimator with the adapted SARAH (Nguyen et al., 2017a) estimator. Pham et al. (2020) show ProxHSPGA also need $\mathcal{O}(\epsilon^{-3})$ trajectories to achieve the ϵ -FOSP. To compare clearly, we summarize more details of the comparison in Table 1.

5.2 Second-Order Measurement

As mentioned in the previous section, for RL, an algorithm converges to a FOSP is not sufficient to ensure that algorithm outputs a maximal point, which is our main motivation to consider SOSP

Algorithm	Conditions	Guarantee	Complexity
REINFORCE (Williams, 1992)	Assumption 1	First-Order	$\mathcal{O}(\epsilon^{-4})$
GPOMDP (Baxter and Bartlett, 2001)	Assumption 1	First-Order	$\mathcal{O}(\epsilon^{-4})$
SVRPG (Papini et al., 2018)	Assumption 1 $\mathbb{V}\mathrm{ar}\Big[\prod_{i\geq 0}\frac{\pi_{\theta_0}(a_i s_i)}{\pi_{\theta_t}(a_i s_i)}\Big] < +\infty$	First-Order	$\mathcal{O}(\epsilon^{-4})$
SVRPG (Xu et al., 2019)	Assumption 1 $\mathbb{V}\mathrm{ar}\bigg[\prod_{i\geq 0}\frac{\pi_{\theta_0}(a_i s_i)}{\pi_{\theta_t}(a_i s_i)}\bigg] < +\infty$	First-Order	$\mathcal{O}(\epsilon^{-rac{10}{3}})$
HAPG (Shen et al., 2019)	Assumption 1	First-Order	$\mathcal{O}(\epsilon^{-3})$
VRMPO (Yang et al., 2019a)	Assumption 1	First-Order	$\mathcal{O}(\epsilon^{-3})$
SRVR-PG (Xu et al., 2020a)	$ \begin{aligned} & \text{Assumption 1} \\ & \mathbb{V}\mathrm{ar} \bigg[\prod_{i \geq 0} \frac{\pi_{\theta_0}(a_i s_i)}{\pi_{\theta_t}(a_i s_i)} \bigg] < +\infty \end{aligned} $	First-Order	${\cal O}(\epsilon^{-3})$
ProxHSPGA (Pham et al., 2020)	$ \begin{aligned} & \text{Assumption } 1 \\ & \mathbb{V}\mathrm{ar}\Big[\prod_{i\geq 0}\frac{\pi_{\theta_0}(a_i s_i)}{\pi_{\theta_t}(a_i s_i)}\Big] < +\infty \end{aligned} $	First-Order	$\mathcal{O}(\epsilon^{-3})$
MRPG (Zhang et al., 2019)	Assumption 1 and Eq. (6)	Second-Order	$\widetilde{\mathcal{O}}ig(\epsilon^{-9}ig)$
Our work	Assumption 1-3	Second-Order	$\widetilde{\mathcal{O}}(\epsilon^{-rac{9}{2}})$

Table 1: Complexity comparison, where the result of first-order requires $\|\nabla J(\theta)\|_2 \leq \epsilon$, section-order requires an additional condition $\lambda_{\max}(\nabla^2 J(\theta)) \leq \sqrt{\chi \epsilon}$.

to measure the convergence of policy gradient method. To the best of our knowledge, Zhang et al. (2019) firstly introduce SOSP to RL to measure the sample complexity of policy gradient methods. Zhang et al. (2019) propose MRPG that needs at least $\tilde{\mathcal{O}}(\epsilon^{-9})$ samples, which is worse than our result $\tilde{\mathcal{O}}(\epsilon^{-\frac{9}{2}})$. We have discussed this comparison in the previous Remark 1.

Additionally, it is noteworthy that although we are all adopting the CNC technique to ensure the local improvement on saddle point region, our technique is different from Zhang et al. (2019) at least from two aspects: Firstly, our CNC condition is more general since we consider the fundamental policy gradient estimator (5) and our analysis can be extended to generalized to extensive policy optimization algorithms; while the CNC result of Zhang et al. (2019) is limited in their proposed algorithm MRPG; Secondly, on the region \mathcal{L}_2 , our result shows that within at most $\mathcal{O}(\epsilon^{-\frac{1}{2}})$ steps, policy gradient ensures an increase in the value of $J(\theta)$. While, Zhang et al. (2019) require $\Omega(\epsilon^{-5} \log \frac{1}{\epsilon})$, which is the main reason why our analysis to achieve a better sample complexity.

5.3 Future Work

In this paper, we mainly consider Monte Carlo gradient estimator (4), the technique of proof can be generalized to extensive policy gradient methods such as replacing $R(\tau)$ with state-action value function $Q^{\pi}(s_t, a_t)$, advantage function $A^{\pi}(s_t, a_t)$, baseline function $R(\tau) - V^{\pi}(s_t, a_t)$, and temporal difference error $r_{t+1} + \gamma V^{\pi}(s_{t+1}, a_{t+1}) - V^{\pi}(s_t, a_t)$.

Our result of $\tilde{\mathcal{O}}(\epsilon^{-\frac{9}{2}})$ to achieve $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP is still far from the best-known ϵ -FOSP result $\mathcal{O}(\epsilon^{-3})$ In theory, Allen-Zhu and Li (2018) and Xu et al. (2018) independently show that finding a SOSP is not much harder than FOSP. Recently, in non-convex optimization, Ge et al. (2019) show that with a simple variant of SVRG, we can find a SOSP that almost matches the known the first-order stationary points.

This provides a motivation that we can introduce some latest developments such as (Du et al., 2017; Daneshmand et al., 2018; Jin et al., 2018; Zhou et al., 2018a,b; Ge et al., 2019; Fang et al., 2019) to give some fresh understanding to RL algorithms. Besides, it will be also interesting to rethink the sample complexity of SOSP of the works (Papini et al., 2018; Shen et al., 2019; Yang et al., 2019a; Pham et al., 2020), where they have proposed SVRG version of policy gradient methods.

It is noteworthy that we don't consider the actor-critic type algorithms. Recently Yang et al. (2019b); Kumar et al. (2019); Agarwal et al. (2019); Xu et al. (2020b); Wang et al. (2020) have analyzed the complexity of actor-critic or natural actor-critic algorithms, and it will be interesting to rethink the sample complexity of SOSP of actor-critic or natural actor-critic algorithms.

6 Conclusion

In this paper, we provide the sample complexity of the policy gradient method finding second-order stationary points. Our result shows that policy gradient methods converge to an $(\epsilon, \sqrt{\epsilon\chi})$ -SOSP at a cost of $\tilde{\mathcal{O}}(\epsilon^{-\frac{9}{2}})$, which improves the the best-known result of by a factor of $\tilde{\mathcal{O}}(\epsilon^{-\frac{9}{2}})$. Besides, we think the technique of proof can be potentially generalized to extensive policy optimization algorithms, and give some fresh understanding to the existing algorithms.

A Table of Notations

For convenience of reference, we list key notations that have be used in this paper.

- \mathcal{S} : The set of states.
- \mathcal{A} : The set of actions.
- P(s'|s, a) : The probability of state transition from s to s' under playing the action a.

 ρ_0 : $\rho_0(\cdot) : \mathcal{S} \to [0,1]$ is the initial state distribution.

- γ : The discount factor, and $\gamma \in (0, 1)$.
- τ : The trajectory generated according to π_{θ} , i.e., $\tau = \{s_t, a_t, r_{t+1}\}_{t \ge 0} \sim \pi_{\theta}$.
- $P^{\pi_{\theta}}(s_t|s_0)$: The probability of visiting the state s_t after t time steps from the initial state s_0 by executing π_{θ} .
 - $d_{s_0}^{\pi_{\theta}}(s)$: The (unnormalized) discounted stationary state distribution of the Markov chain (starting at s_0) induced by π_{θ} , $d_{s_0}^{\pi_{\theta}}(s) = \sum_{t=0}^{\infty} \gamma^t P^{\pi_{\theta}}(s_t = s | s_0)$.

$$d^{\pi_{\theta}}_{\rho_{0}}(s)$$
 : $d^{\pi_{\theta}}_{\rho_{0}}(s) = \mathbb{E}_{s_{0} \sim \rho_{0}(\cdot)}[d^{\pi_{\theta}}_{s_{0}}(s)]$

- $J(\theta)$: The performance objective, defined in (1).
- $g(\tau|\theta)$: An policy gradient estimator $g(\tau|\theta) = \sum_{t=0}^{\infty} \nabla \log \pi_{\theta}(a_t|s_t) R(\tau)$, defined in (4).
- $F(\theta)$: Fisher information matrix defined in (6).
- ω : A positive scaler defined in (6).

$$G,L,U$$
 : Two positive scalers defined in (7).

- ℓ : Lipschitz parameter of $\nabla J(\theta)$, and it is defined in (8).
- σ : An upper-bound of $\mathbb{V}ar(g(\tau|\theta))$, and it is defined in (9).
- χ : Hessian-Lipschitz parameter χ presented in (10).

B Some Lemmas

Lemma 1 (Azuma's Inequality). Let $\{Z_t\}_{t\in\mathbb{N}}$ be a martingale with respect to the filtration: $\mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_t \subset \cdots$. Assume that there are predictable processes $\{A_t\}_{t\in\mathbb{N}}$ and $\{B_t\}_{t\in\mathbb{N}}$ (i.e., $A_t, B_t \in \mathcal{F}_{t-1}$) and constants $0 < c_t < +\infty$ such that: for all $t \geq 1$, almost surely,

$$A_t \leq Z_t - Z_{t-1} \leq B_t$$
 and $B_t - A_t \leq c_t$.

Then, for all $\delta > 0$,

$$\mathbb{P}[Z_t - Z_0 \ge \delta] \le \exp\left(-\frac{2\delta^2}{\sum_{i=0}^t c_i^2}\right).$$

For the proof of Azuma's inequality, please refer to http://www.math.wisc.edu/~roch/grad-prob/gradprob-note

The following Lemma 2 illustrates the difference of the performance between two policies and it is helpful throughout this paper.

Lemma 2. ((Kakade and Langford, 2002)) For any policy π , $\tilde{\pi}$, and any initial state $s_0 \in S$, we have

$$V^{\pi}(s_0) - V^{\tilde{\pi}}(s_0) = \mathbb{E}_{s \sim d_{s_0}^{\pi}(\cdot), a \sim \pi(\cdot|s)} [A^{\tilde{\pi}}(s, a)].$$
(26)

Lemma 3 ((Zhang et al., 2019)). Under Assumption 1. With an additional condition as follows,

$$\|\nabla^2 \log \pi_{\theta_1}(a|s) - \nabla^2 \log \pi_{\theta_2}(a|s)\|_2 \le W \|\theta_1 - \theta_2\|_2,$$

then we have estimate of χ such that

$$\|\nabla^2 J(\theta) - \nabla^2 J(\theta')\|_{op} \le \chi \|\theta - \theta'\|_2,$$

where

$$\chi =: \frac{R_{\max}GL}{(1-\gamma)^2} + \frac{R_{\max}G^3(1+\gamma)}{(1-\gamma)^3} + \frac{R_{\max}G}{1-\gamma} \max\left\{L, \frac{\gamma G^2}{1-\gamma}, \frac{W}{G}, \frac{L\gamma}{1-\gamma}, \frac{G(1+\gamma) + L\gamma(1-\gamma)}{1-\gamma^2}\right\}.$$

Remark 2. Lemma 3 illustrates, with some other regularity conditions, we can give a concrete estimate of the parameter χ that satisfies Assumption 2. This Assumption 2 significantly simplifies the theory analysis, but it could be removed by other regularity conditions.

C Proof of Proposition 1

Proposition 1 Under Assumption 1-3. The sequence $\{\theta_k\}_{k\geq 0}$ generated according to (5). If a point $\theta_k \in \mathcal{L}_1$, i.e., $\left\|\nabla J(\theta_k)\right\|_2 \geq \epsilon$, and let $\alpha < \left\{\frac{2\epsilon^2}{(\epsilon^2 + \sigma^2)\ell}, \frac{2}{\ell}\right\}$, then after one update, the following holds

$$\mathbb{E}\left[J(\theta_{k+1})\right] - J(\theta_k) \ge \left(\alpha - \frac{\ell\alpha^2}{2}\right) \left\|\nabla J(\theta_k)\right\|_2^2 - \frac{\ell\alpha^2\sigma^2}{2}$$
$$\ge \alpha\epsilon^2 - \frac{1}{2}\ell\alpha^2(\epsilon^2 + \sigma^2) > 0.$$

Proof. (of Proposition 1)

Recall for each $k \in \mathbb{N}$, $\tau_k \sim \pi_{\theta_k}$, to simplify expression, we introduce a notation as follows,

$$g(\theta_k) =: g(\theta_k | \tau_k) = \sum_{t \ge 0} \nabla_\theta \log \pi_\theta(a_t | s_t) R(\tau_k) |_{\theta = \theta_k}$$

Then, the update (5) can be rewritten as follows,

$$\theta_{k+1} = \theta_k + \alpha g(\theta_k).$$

Let

$$\xi_k = g(\theta_k) - \nabla J(\theta_k), \tag{27}$$

By the fact $\nabla J(\theta_k) = \mathbb{E}[g(\theta_k)]$, we have $\mathbb{E}[\xi_k] = 0$. By the result of (8), i.e., for each $\theta, \theta', \|\nabla J(\theta) - \nabla J(\theta')\|_2 \le \ell \|\theta - \theta'\|_2$, which implies

$$|J(\theta') - J(\theta) - \langle \nabla J(\theta), \theta' - \theta \rangle| \le \frac{\ell}{2} \left\| \theta - \theta' \right\|_2^2.$$
⁽²⁸⁾

Then, we have

$$J(\theta_{k+1}) - J(\theta_k) \ge \langle \nabla J(\theta_k), \theta_{k+1} - \theta_k \rangle - \frac{\ell}{2} \left\| \theta_k - \theta_{k+1} \right\|_2^2,$$

which implies

$$\begin{split} \mathbb{E} \left[J(\theta_{k+1}) \right] - J(\theta_k) &\geq \nabla J(\theta_k)^\top \mathbb{E} \left[\theta_{k+1} - \theta_k \right] - \frac{\ell}{2} \mathbb{E} \left[\left\| \theta_k - \theta_{k+1} \right\|_2^2 \right] \\ &= \nabla J(\theta_k)^\top \mathbb{E} \left[\alpha g(\theta_k) \right] - \frac{\ell \alpha^2}{2} \mathbb{E} \left[\left\| \xi_0^\top \xi_0 + \nabla J(\theta_k) \right\|_2^2 \right] \\ &= \nabla J(\theta_k)^\top \mathbb{E} \left[\alpha g(\theta_k) \right] - \frac{\ell \alpha^2}{2} \mathbb{E} \left[\left\| \xi_0^\top \xi_0 + 2\xi_0^\top \nabla J(\theta_k) + \nabla J(\theta_k)^\top \nabla J(\theta_k) \right] \\ &= \left(\alpha - \frac{\ell \alpha^2}{2} \right) \left\| \nabla J(\theta_k) \right\|_2^2 - \frac{\ell \alpha^2}{2} \mathbb{E} \left[\underbrace{\xi_0^\top \xi_0}_{=} \right] \\ &= \left\| \nabla J(\theta_k) - g(\theta_k) \right\|_2^2 \\ &\stackrel{(9)}{\geq} \left(\alpha - \frac{\ell \alpha^2}{2} \right) \left\| \nabla J(\theta_k) \right\|_2^2 - \frac{\ell \alpha^2 \sigma^2}{2} \\ &\geq \left(\alpha - \frac{\ell \alpha^2}{2} \right) \epsilon^2 - \frac{\ell \alpha^2 \sigma^2}{2} = \alpha \epsilon^2 - \frac{1}{2} \ell \alpha^2 \epsilon^2 - \frac{1}{2} \ell \alpha^2 \sigma^2, \end{split}$$

Solving
$$\frac{1}{2}\alpha\epsilon^2 - \frac{1}{2}\ell\alpha^2\epsilon^2 - \frac{1}{2}\ell\alpha^2\sigma^2 > 0$$
, we have $\alpha < \frac{\epsilon^2}{(\epsilon^2 + \sigma^2)\ell}$, then we have $\mathbb{E}[J(\theta_{k+1})] - J(\theta_k) \ge \frac{1}{2}\alpha\epsilon^2$.

This concludes the proof.

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D Proof of Proposition 3

In this section we provide the details of the proof of Proposition 3. We need the following Lemma 4, Lemma 5 and Lemma 6 to show the result of Proposition 3.

Proposition 3 Under Assumption 1-3, Consider the sequence $\{\theta_k\}_{k\geq 0}$ generated by (5). For any $\delta \in (0,1)$, let the step-size α and the stopping time κ_0 statisfy

$$\alpha \le \min\left\{\delta, \frac{1}{\zeta}, \frac{\zeta}{\ell^2}, \frac{\zeta\varrho^2}{3\sigma^2}\right\}, \quad \alpha \log\frac{1}{\alpha} \le \frac{2\zeta\varrho^4}{27\left(\frac{G^2R_{\max}^2}{(1-\gamma)^2} + \zeta\varrho^2 + \sigma^2\right)^2}, \quad \kappa_0 = \left\lfloor\frac{1}{\alpha^2}\log\frac{1}{\delta}\right\rfloor$$

and if some iteration θ_k falls into the ball $\mathbb{B}_2(\theta_\star, \frac{\sqrt{3}}{3}\varrho) \subset \mathbb{B}_2(\theta_\star, \varrho)$, i.e., $\|\theta_k - \theta_\star\|_2^2 \leq \frac{1}{3}\varrho^2$. Then, with probability at least $1 - \delta \log \frac{1}{\delta}$, for all $j \in [0, \kappa_0 - 1]$, we have $\|\theta_{k+j} - \theta_\star\|_2 \leq \varrho$, i.e.,

$$\mathbb{P}\Big(\big\|\theta_{k+j} - \theta_\star\big\|_2 \le \varrho\Big) \ge 1 - \delta \log \frac{1}{\delta}.$$

Recall for each $k \in \mathbb{N}$, $\tau_k \sim \pi_{\theta_k}$, to simplify expression, we introduce a notation as follows,

$$g(\theta_k) =: g(\theta_k | \tau_k) = \sum_{t=0}^h \nabla_\theta \log \pi_\theta(a_t | s_t) R(\tau_k) |_{\theta = \theta_k}$$

Then, the update (5) can be rewritten as follows,

$$\theta_{k+1} = \theta_k + \alpha g(\theta_k).$$

Lemma 4 (Boundedness of the near iteration). Consider the sequence $\{\theta_k\}_{k\in\mathbb{N}}$ generated according to (5), i.e., $\theta_{k+1} = \theta_k + \alpha g(\theta_k)$. Then,

$$\left\|\theta_{k+1} - \theta_k\right\|_2 \le \alpha \left\|g(\theta_{k+1}) - g(\theta_k)\right\|_2.$$
⁽²⁹⁾

,

Lemma 4 is a direct result of Lemma 2 in (Ghadimi et al., 2016). We omit its proof.

We use $\mathcal{E}_{[k:k+t]}$ to denote the event that the element of the sequence $\{\theta_j\}_{j=k}^{k+t}$ falls into the ball $\mathbb{B}_2(\theta_\star, \varrho)$, and use κ_0 to denote its stopping time, i.e.,

$$\mathcal{E}_{[k:k+t]} = \bigcap_{j=0}^{t} \left\{ \theta_{k+j} : \left\| \theta_{k+j} - \theta_{\star} \right\|_{2} \le \varrho \right\},$$

$$\kappa_{0} = \inf_{j \ge 0} \left\{ j : \left\| \theta_{k+j} - \theta_{\star} \right\|_{2} > \varrho \right\}.$$
(30)

From the definition of (30), we notice two following basic facts:

• If $t > \kappa_0$, the event $\mathcal{E}_{[k:k+t]}$ never happens, i.e., $\mathcal{E}_{[k:k+t]} = \emptyset$ after the time κ_0 ; For each $0 \le j < \kappa_0$, the point θ_{k+j} falls into the ball $\mathbb{B}_2(\theta_\star, \varrho)$.

2 For each t, we have $\mathcal{E}_{[k:k+t+1]} \subset \mathcal{E}_{[k:k+t]}$.

Lemma 5. We consider the term $Z_j = \max\left\{(1 - \alpha\zeta)^{-j}\left(\left\|\theta_{k+j} - \theta_{\star}\right\|_2^2 - \frac{\alpha\sigma^2}{\zeta}\right), 0\right\}$, then for each $j \in [0, \kappa_0 - 1]$, the following holds

$$|Z_{j+1} - Z_j| \le \underbrace{(1 - \alpha\zeta)^{-(j+1)} \left(\alpha^2 2G^2 R_{\max}^2 / (1 - \gamma)^2 + \alpha\zeta\varrho^2 + \alpha^2\sigma^2 \right)}_{=:c_{j+1}},\tag{31}$$

i.e., we have $|Z_{j+1} - Z_j| \le c_{j+1}$.

Lemma 5 illustrates the boundedness of $Z_{j+1} - Z_j$, which is helpful for us to use Azuma's inequality to achieve more refined results later.

Lemma 6. Consider the sequence $\{\theta_k\}_{k\in\mathbb{N}}$ generated according to (5), i.e., $\theta_{k+1} = \theta_k + \alpha g(\theta_k)$. Then, for the term $\theta_{k+\kappa_0} - \theta_{\star}$, the following holds

$$\mathbb{P}\Big(\left\|\theta_{k+\kappa_0-1}-\theta_\star\right\|_2^2 \ge \varrho^2\Big) \le \alpha^2 \delta.$$

D.1 Some Preliminary Results

Before we give the details of the proof of Lemma 5, Lemma 6 and Proposition 3, we present some preliminary results.

Let

$$\xi_k = g(\theta_k) - \nabla J(\theta_k), \tag{32}$$

By the fact $\nabla J(\theta_k) = \mathbb{E}[g(\theta_k)]$, we have $\mathbb{E}[\xi_k] = 0$.

We use $\mathcal{F}_{[k,k+t]} = \sigma(\xi_k, \cdots, \xi_{k+t})$ to denote the σ -field generated by the information from time k to k + t: $\{\xi_k, \cdots, \xi_{k+t}\}$.

Recall the update rule (5), let $\alpha_k = \alpha \leq \frac{\zeta}{\ell^2}$, we analyze the $(k + \kappa_0)$ -th iteration as follows,

$$\theta_{k+\kappa_0} = \theta_{k+\kappa_0-1} + \alpha g(\theta_{k+\kappa_0-1})$$

$$= \theta_{k+\kappa_0-1} + \alpha \left(\underbrace{g(\theta_{k+\kappa_0-1}) - \nabla J(\theta_{k+\kappa_0-1})}_{=\xi_{k+\kappa_0-1}; \text{Eq.}(32)} + \nabla J(\theta_{k+\kappa_0-1}) \right),$$

$$= \theta_{k+\kappa_0-1} + \alpha \left(\nabla J(\theta_{k+\kappa_0-1}) + \xi_{k+\kappa_0-1} \right).$$
(33)

Under Assumption 3, $J(\theta)$ is ζ -strongly concave in $\mathbb{B}(\theta_{\star}, \varrho)$, i.e.,

$$\nabla J(\theta') \le J(\theta) + \nabla J(\theta)^{\top}(\theta' - \theta) - \frac{\zeta}{2} \|\theta' - \theta\|_2^2, \quad \forall \ \theta, \theta' \in \mathbb{B}_2(\theta_{\star}, \varrho).$$
(34)

Since θ_k falls into the ball $\mathbb{B}_2(\theta_\star, \frac{\sqrt{3}}{3}\varrho) \subset \mathbb{B}_2(\theta_\star, \varrho)$, recall $\kappa_0 = \inf_{j\geq 0} \{j : \|\theta_{k+j} - \theta_\star\|_2 > \varrho\}$, thus the point $\theta_{k+\kappa_0-1}$ also falls into the ball $\mathbb{B}(\theta_\star, \varrho)$, according to (34), we have

$$J(\theta_{\star}) - J(\theta_{k+\kappa_{0}-1}) \leq \nabla J(\theta_{k+\kappa_{0}-1})^{\top} (\theta_{\star} - \theta_{k+\kappa_{0}-1}) - \frac{\zeta}{2} \|\theta_{k+\kappa_{0}-1} - \theta_{\star}\|_{2}^{2}$$
$$J(\theta_{k+\kappa_{0}-1}) - J(\theta_{\star}) \leq \underbrace{\nabla J(\theta_{\star})^{\top}}_{=0} (\theta_{k+\kappa_{0}-1} - \theta_{\star}) - \frac{\zeta}{2} \|\theta_{\star} - \theta_{k+\kappa_{0}-1}\|_{2}^{2},$$

which implies

$$\zeta \left\| \theta_{k+\kappa_0-1} - \theta_\star \right\|_2^2 \le \nabla J \left(\theta_{k+\kappa_0-1} \right)^\top \left(\theta_\star - \theta_{k+\kappa_0-1} \right).$$
(35)

Under Assumption 1, recall the result of (8), we have

$$\left\|\nabla J(\theta_{k+\kappa_0-1}) - \nabla J(\theta_{\star})\right\|_2 = \left\|\nabla J(\theta_{k+\kappa_0-1})\right\|_2 \le \ell \left\|\theta_{k+\kappa_0-1} - \theta_{\star}\right\|_2.$$
(36)

D.2 Proof of Lemma 5

Proof. (of Lemma 5).

Recall $\mathcal{F}_{[k,k+t]} = \sigma(\xi_k, \cdots, \xi_{k+t})$ is the σ -field generated by the information from time k to k + t. Firstly, we turn to analyze the expected gap between $\theta_{k+\kappa_0}$ and θ_{\star} as follows,

$$\mathbb{E} \left[\left\| \theta_{k+\kappa_{0}} - \theta_{\star} \right\|_{2}^{2} \left| \mathcal{F}_{[k:k+\kappa_{0}-1]} \right] \right] \\
\stackrel{(33)}{=} \mathbb{E} \left[\left\| \theta_{k+\kappa_{0}-1} + \alpha \left(\nabla J(\theta_{k+\kappa_{0}-1}) + \xi_{k+\kappa_{0}-1} \right) - \theta_{\star} \right\|_{2}^{2} \left| \mathcal{F}_{[k:k+\kappa_{0}-1]} \right] \right] \\
= \left\| \theta_{k+\kappa_{0}-1} - \theta_{\star} \right\|_{2}^{2} + \alpha^{2} \underbrace{\mathbb{E}} \left[\left\| \xi_{k+\kappa_{0}-1} \right\|_{2}^{2} \left| \mathcal{F}_{[k:k+\kappa_{0}-1]} \right| \right] \\
\stackrel{(36)}{= \sigma^{2}; \text{Eq.}(9)} \\
+ \alpha^{2} \underbrace{\left\| \nabla J(\theta_{k+\kappa_{0}-1}) \right\|_{2}^{2}}_{(\frac{36}{\leq} \ell^{2} \left\| \theta_{k+\kappa_{0}-1} - \theta_{\star} \right\|_{2}^{2}} + 2\alpha \underbrace{\left\{ \nabla J(\theta_{k+\kappa_{0}-1}), \theta_{k+\kappa_{0}-1} - \theta_{\star} \right\} \right]}_{(\frac{35}{\leq} -\zeta \left\| \theta_{k+\kappa_{0}-1} - \theta_{\star} \right\|_{2}^{2}} \\ \leq (1 + \alpha^{2}\ell^{2} - 2\alpha\zeta) \left\| \theta_{k+\kappa_{0}-1} - \theta_{\star} \right\|_{2}^{2} + \alpha^{2}\sigma^{2} \tag{37}$$

$$\leq (1 + \alpha^{-} \ell^{-} - 2\alpha\zeta) \|\theta_{k+\kappa_{0}-1} - \theta_{\star}\|_{2}^{2} + \alpha^{2} \sigma^{2}$$

$$\leq (1 - \alpha\zeta) \|\theta_{k+\kappa_{0}-1} - \theta_{\star}\|_{2}^{2} + \alpha^{2} \sigma^{2},$$
(37)
(37)
(37)

Eq.(38) holds since $\alpha \leq \frac{\zeta}{\ell^2}$.

Furthermore, rearranging Eq.(38), we have

$$\mathbb{E}\Big[\Big\|\theta_{k+\kappa_0} - \theta_\star\Big\|_2^2\Big|\mathcal{F}_{[k:k+\kappa_0-1]}\Big] - \frac{\alpha\sigma^2}{\zeta} \le (1 - \alpha\zeta)\Big(\Big\|\theta_{k+\kappa_0-1} - \theta_\star\Big\|_2^2 - \frac{\alpha\sigma^2}{\zeta}\Big). \tag{39}$$

Now, , we define a sequence $\{Z_j\}_{j=0}^{\kappa_0}$ with its element as follows,

$$Z_j = \max\left\{ (1 - \alpha\zeta)^{-j} \left(\left\| \theta_{k+j} - \theta_\star \right\|_2^2 - \frac{\alpha\sigma^2}{\zeta} \right), 0 \right\}; \quad 0 \le j \le \kappa_0.$$

$$\tag{40}$$

Since $\kappa_0 = \inf_{j\geq 0} \{j : \|\theta_{k+j} - \theta_\star\|_2 > \varrho\}$, recall $\mathcal{E}_{[k:k+t]} = \bigcap_{j=0}^t \{\theta_{k+j} : \|\theta_{k+j} - \theta_\star\|_2 \le \varrho\}$, which implies $\mathbf{1}_{\mathcal{E}_{[k:k+j]}} = 1$, for each $j \in [0, \kappa_0 - 1]$. Thus,

$$Z_{j+1}\mathbf{1}_{\mathcal{E}_{[k:k+j]}} = Z_{j+1}.$$
(41)

Furthermore, from the result of (39), $Z_{j+1} \mathbf{1}_{\mathcal{E}_{[k:k+j]}}$ is a super-martingale, i.e.,

$$\mathbb{E}\Big[Z_{j+1}\mathbf{1}_{\mathcal{E}_{[k:k+j]}}\Big] \le Z_j\mathbf{1}_{\mathcal{E}_{[k:k+j-1]}}.$$
(42)

Let's bound on the term $|Z_{j+1} - Z_j| \stackrel{(41)}{=} |Z_{j+1} \mathbf{1}_{\mathcal{E}_{[k:k+j]}} - Z_j \mathbf{1}_{\mathcal{E}_{[k:k+j-1]}}|$, for each $0 \le j \le \kappa_0 - 1$, the following holds

$$\begin{aligned} |Z_{j+1} - Z_{j}| \\ \leq \left| (1 - \alpha\zeta)^{-(j+1)} \left(\left(\left\| \theta_{k+j+1} - \theta_{\star} \right\|_{2}^{2} - \frac{\alpha\sigma^{2}}{\zeta} \right) - (1 - \alpha\zeta) \left(\left\| \theta_{k+j} - \theta_{\star} \right\|_{2}^{2} - \frac{\alpha\sigma^{2}}{\zeta} \right) \right) \right| \\ = (1 - \alpha\zeta)^{-(j+1)} \left| \left\| \theta_{k+j+1} - \theta_{\star} \right\|_{2}^{2} - \left\| \theta_{k+j} - \theta_{\star} \right\|_{2}^{2} + \alpha\zeta \left\| \theta_{k+j} - \theta_{\star} \right\|_{2}^{2} - \alpha^{2}\sigma^{2} \right| \\ \leq (1 - \alpha\zeta)^{-(j+1)} \left(\underbrace{\left\| \theta_{k+j+1} - \theta_{\star} \right\|_{2}^{2} - \left\| \theta_{k+j} - \theta_{\star} \right\|_{2}^{2}}_{\leq \left\| \theta_{k+j+1} - \theta_{k+j} \right\|_{2}^{2}} + \alpha\zeta \underbrace{\left\| \theta_{k+j} - \theta_{\star} \right\|_{2}^{2}}_{\leq \varrho^{2}} + \alpha^{2}\sigma^{2} \right) \\ \leq (1 - \alpha\zeta)^{-(j+1)} \left(\underbrace{\left\| \theta_{k+j+1} - \theta_{k+j} \right\|_{2}^{2}}_{\leq \left\| \theta_{k+j} - \theta_{\star} \right\|_{2}^{2}} + \alpha\zeta \underbrace{\left\| \theta_{k+j} - \theta_{\star} \right\|_{2}^{2}}_{\leq \varrho^{2}} + \alpha^{2}\sigma^{2} \right) \end{aligned}$$
(43)

Under Assumption 1, $\{g(\theta_k)\}_{k\geq 0}$ is uniformly bounded, i.e., for each k, we have

$$\left\|g(\theta_k)\right\|_2 = \left\|\sum_{t\geq 0} \underbrace{\nabla_{\theta} \log \pi_{\theta_k}(a_t|s_t)}_{\leq G;(7)} R(\tau_k)\right\|_2 \leq G \sum_{t\geq 0} \gamma^t r_{t+1} \leq \frac{GR_{\max}}{1-\gamma}.$$
(44)

Thus, we can rewrite (43) as follows,

$$|Z_{j+1} - Z_j| \le (1 - \alpha\zeta)^{-(j+1)} \left(\alpha^2 \|g(\theta_{k+j+1})\|_2^2 + \alpha^2 \|g(\theta_{k+j})\|_2^2 + \alpha\zeta\varrho^2 + \alpha^2\sigma^2 \right)$$

$$\stackrel{(44)}{\le} (1 - \alpha\zeta)^{-(j+1)} \left(\alpha^2 \frac{2G^2 R_{\max}^2}{(1 - \gamma)^2} + \alpha\zeta\varrho^2 + \alpha^2\sigma^2 \right) =: c_{j+1}.$$
(45)

This concludes the proof.

D.3 Proof of Lemma 6

Proof. (of Lemma 6). Recall the results of (42) and (31). By Azuma's inequality, consider the sequence $\{Z_j\}_{j=0}^{\kappa_0-1}$ defined in Eq.(40), recall c_j is defined in (45), for any $\delta > 0$, we have

$$\mathbb{P}\Big(Z_{\kappa_0-1}-Z_0 \ge \delta\Big) \le \exp\Big(-\frac{2\delta^2}{\sum_{j=0}^{\kappa_0-1} c_j^2}\Big),$$

which implies the following holds,

$$\mathbb{P}\left(Z_{\kappa_0-1} - Z_0 \ge \sqrt{\frac{1}{2}\sum_{j=0}^{\kappa_0-1} c_j^2 \left(\log\frac{1}{\delta} + 2\log\frac{1}{\alpha}\right)}\right) \le \alpha^2 \delta.$$
(46)

The result of (46) implies the following holds with probability less than $\alpha^2 \delta$,

$$\begin{aligned} \left\|\theta_{k+\kappa_{0}-1}-\theta_{\star}\right\|_{2}^{2} &-\frac{\alpha\sigma^{2}}{\zeta} \geq (1-\alpha\zeta)^{\kappa_{0}-1}\sqrt{\frac{1}{2}\sum_{j=0}^{\kappa_{0}-1}c_{j}^{2}\left(\log\frac{1}{\delta}+2\log\frac{1}{\alpha}\right)} \\ &+(1-\alpha\zeta)^{\kappa_{0}-1}\left(\left\|\theta_{k}-\theta_{\star}\right\|_{2}^{2}-\frac{\alpha\sigma^{2}}{\zeta}\right). \end{aligned}$$
(47)

Rearranging Eq.(47), we have

$$\|\theta_{k+\kappa_{0}-1} - \theta_{\star}\|_{2}^{2} \geq \underbrace{(1 - \alpha\zeta)^{\kappa_{0}-1} \sqrt{\frac{1}{2} \sum_{j=0}^{\kappa_{0}-1} c_{j}^{2} \left(\log \frac{1}{\delta} + 2\log \frac{1}{\alpha}\right)}_{I_{1}} + \underbrace{(1 - \alpha\zeta)^{\kappa_{0}-1} \|\theta_{k} - \theta_{\star}\|_{2}^{2}}_{I_{2}} + \underbrace{(1 - (1 - \alpha\zeta)^{\kappa_{0}-1}) \frac{\alpha\sigma^{2}}{\zeta}}_{I_{3}}.$$

$$(48)$$

We need to give the conditions to achieve the boundedness of the terms I_1, I_2, I_3 defined in Eq.(48). <u>Boundedness of I_1 in Eq.(48)</u>. Consider c_j in (45), i.e., $c_j = (1 - \alpha\zeta)^{-j} \left(\alpha^2 2G^2 R_{\max}^2 / (1 - \gamma)^2 + \alpha\zeta \varrho^2 + \alpha^2 \sigma^2 \right)$, and with the condition $\alpha \leq \delta$, we have

$$\begin{split} I_{1} &= (1 - \alpha\zeta)^{\kappa_{0}-1} \sqrt{\frac{1}{2} \sum_{j=0}^{\kappa_{0}-1} c_{j}^{2} \log \frac{1}{\delta}} \\ &= \left(\alpha^{2} \frac{2G^{2} R_{\max}^{2}}{(1 - \gamma)^{2}} + \alpha\zeta\varrho^{2} + \alpha^{2}\sigma^{2}\right) \sqrt{\frac{1}{2} \sum_{j=0}^{\kappa_{0}-1} (1 - \alpha\zeta)^{2(\kappa_{0}-1)-2j}} \sqrt{\log \frac{1}{\delta}} + 2\log \frac{1}{\alpha}} \\ &\leq \left(\alpha^{2} \frac{2G^{2} R_{\max}^{2}}{(1 - \gamma)^{2}} + \alpha\zeta\varrho^{2} + \alpha^{2}\sigma^{2}\right) \sqrt{\frac{1}{2} \sum_{j=0}^{\kappa_{0}-1} (1 - \alpha\zeta)^{2j}} \sqrt{3\log \frac{1}{\alpha}} \\ &\leq \left(\alpha^{2} \frac{2G^{2} R_{\max}^{2}}{(1 - \gamma)^{2}} + \alpha\zeta\varrho^{2} + \alpha^{2}\sigma^{2}\right) \sqrt{\frac{1}{2} \sum_{j=0}^{\infty} (1 - \alpha\zeta)^{2j}} \sqrt{3\log \frac{1}{\alpha}} \\ &= \left(\alpha^{2} \frac{2G^{2} R_{\max}^{2}}{(1 - \gamma)^{2}} + \alpha\zeta\varrho^{2} + \alpha^{2}\sigma^{2}\right) \sqrt{\frac{3\log \frac{1}{\alpha}}{2\left(1 - (1 - \alpha\zeta)^{2}\right)}} \\ &\leq \frac{\left(\alpha^{2} \frac{2G^{2} R_{\max}^{2}}{(1 - \gamma)^{2}} + \alpha\zeta\varrho^{2} + \alpha^{2}\sigma^{2}\right)}{\sqrt{2\alpha\zeta}} \sqrt{3\log \frac{1}{\alpha}} \\ &\leq \frac{\left(\alpha^{2} \frac{2G^{2} R_{\max}^{2}}{(1 - \gamma)^{2}} + \alpha\zeta\varrho^{2} + \alpha^{2}\sigma^{2}\right)}{\sqrt{2\alpha\zeta}} \sqrt{3\log \frac{1}{\alpha}}$$

$$\end{split} \tag{49}$$

where Eq.(49) holds since $\alpha \zeta \in (0, 1)$, then $1 - (1 - \alpha \zeta)^2 = 2\alpha \zeta - (\alpha \zeta)^2 = \alpha \zeta (2 - \alpha \zeta) \ge \alpha \zeta$; Eq.(50) holds since $\alpha \in (0, 1)$.

We turn to find the condition satisfies

$$I_1 = \frac{\sqrt{\alpha} \left(\frac{G^2 R_{\max}^2}{(1-\gamma)^2} + \zeta \varrho^2 + \sigma^2\right)}{\sqrt{2\zeta}} \sqrt{3\log\frac{1}{\alpha}} \le \frac{1}{3}\varrho^2,\tag{51}$$

which requires the step-size should stisfy

$$\alpha \log \frac{1}{\alpha} \le \frac{2\zeta \varrho^4}{27 \left(\frac{G^2 R_{\max}^2}{(1-\gamma)^2} + \zeta \varrho^2 + \sigma^2\right)^2}.$$
(52)

It is noteworthy that since $\lim_{\alpha\to 0^+} \alpha \log \frac{1}{\alpha} = 0$, then for an enough small α , the condition (52) alway holds.

Boundedness of I_2 in Eq.(48). In fact, $I_2 = (1 - \alpha \zeta)^{\kappa_0 - 1} \|\theta_k - \theta_\star\|_2^2 \le \|\theta_k - \theta_\star\|_2^2 \le \frac{1}{3}\varrho^2$.

Boundedness of I_3 in Eq.(48). In fact, let's consider $I_3 = (1 - (1 - \alpha\zeta)^{\kappa_0 - 1})\frac{\alpha\sigma^2}{\zeta} \le \frac{\alpha\sigma^2}{\zeta} \le \frac{1}{3}\varrho^2$, which implies we need the following condition of step-size: $\alpha \le \frac{\zeta\varrho^2}{3\sigma^2}$.

By the results of all the above boundedness of I_1, I_2, I_3 , under the following condition of step-size

$$\alpha \le \min\left\{\delta, \frac{1}{\zeta}, \frac{\zeta}{\ell^2}, \frac{\zeta\varrho^2}{3\sigma^2}\right\}, \quad \alpha \log\frac{1}{\alpha} \le \frac{2\zeta\varrho^4}{27\left(G^2 R_{\max}^2/(1-\gamma)^2 + \zeta\varrho^2 + \sigma^2\right)^2}$$

the result of (48) implies

$$\mathbb{P}\Big(\big\|\theta_{k+\kappa_0-1}-\theta_\star\big\|_2^2 \ge \varrho^2\Big) \le \alpha^2\delta$$

This concludes the proof.

D.4 Proof of Proposition 3

Proof. (of Proposition 3) Let $\overline{\mathcal{E}_{[k:k+t]}}$ be the complementary event of $\mathcal{E}_{[k:k+t]}$. Recall

$$\mathcal{E}_{[k:k+t]} = \bigcap_{j=0}^{t} \Big\{ \theta_{k+j} : \big\| \theta_{k+j} - \theta_{\star} \big\|_{2} \le \varrho \Big\},$$

according to the basic law of set theory, we have

$$\overline{\mathcal{E}_{[k:k+\kappa_0]}} = \overline{\bigcap_{j=0}^{\kappa_0} \left\{ \theta_{k+j} : \left\| \theta_{k+j} - \theta_\star \right\|_2 \le \varrho \right\}}$$

$$= \overline{\bigcup_{j=0}^{\kappa_0} \left\{ \left\| \theta_{k+j} - \theta_\star \right\|_2 \le \varrho \right\}} = \overline{\bigcup_{j=0}^{\kappa_0} \left\{ \left\| \theta_{k+j} - \theta_\star \right\|_2 > \varrho \right\}}$$

$$= \overline{\bigcup_{j=0}^{\kappa_0 - 1} \left\{ \left\| \theta_{k+j} - \theta_\star \right\|_2 > \varrho \right\}} \bigcup \left\{ \left\| \theta_{k+\kappa_0} - \theta_\star \right\|_2 > \varrho \right\}.$$
(53)

Then,

$$\mathbb{P}\left(\overline{\mathcal{E}_{[k:k+\kappa_0]}}\right) \le \mathbb{P}\left(\bigcup_{j=0}^{\kappa_0-1} \left\{ \left\|\theta_{k+j} - \theta_\star\right\|_2 > \varrho \right\} \right) + \mathbb{P}\left(\left\|\theta_{k+\kappa_0} - \theta_\star\right\|_2 > \varrho \right) \le \mathbb{P}\left(\overline{\mathcal{E}_{[k:k+\kappa_0-1]}}\right) + \alpha^2 \delta.$$
(54)

Furthermore, the the condition θ_k falls in the ball $\mathbb{B}_2(\theta_\star, \frac{\sqrt{3}}{3}\varrho)$ implies

$$\mathbb{P}\Big(\overline{\mathcal{E}_{[k:k]}}\Big) = 0;$$

By the result of (54), we have

$$\mathbb{P}\left(\overline{\mathcal{E}_{[k:k+\kappa_0]}}\right) = \sum_{j=1}^{\kappa_0} \left(\mathbb{P}\left(\overline{\mathcal{E}_{[k:k+j]}}\right) - \mathbb{P}\left(\overline{\mathcal{E}_{[k:k+j-1]}}\right)\right) + \mathbb{P}\left(\overline{\mathcal{E}_{[k:k]}}\right) \le \kappa_0 \alpha^2 \delta.$$
(55)

Let $\kappa_0 = \left\lfloor \frac{1}{\alpha^2} \log \frac{1}{\delta} \right\rfloor + 1$, then we have $\mathbb{P}\left[\overline{\mathcal{E}_{[k:k+\kappa_0]}}\right] \leq \delta \log \frac{1}{\delta}$, which implies for each $0 \leq j \leq \left\lfloor \frac{1}{\alpha^2} \log \frac{1}{\delta} \right\rfloor + 1$, the following holds with probability at least $1 - \delta \log \frac{1}{\delta}$

$$\left\|\theta_{k+j} - \theta_{\star}\right\|_{2}^{2} \le \varrho^{2},\tag{56}$$

i.e., for each $j \in [0, \kappa_0]$, we have

$$\mathbb{P}\Big(\big\|\theta_{k+j} - \theta_\star\big\|_2 \le \varrho\Big) \ge 1 - \delta \log \frac{1}{\delta}.$$

This concludes the proof.

E Proof of Proposition 2

Proposition 2 Under Assumption 1-3, consider the sequence $\{\theta_k\}_{k\geq 0}$ generated by (5). If a point θ_k satisfies

$$\lambda_{\max}(\nabla^2 J(\theta_k)) \ge \sqrt{\chi \epsilon}, \|\nabla J(\theta_k)\|_2 \le \epsilon,$$

there exists a positive scalar $\hat{\kappa}_0$:

$$\widehat{\kappa}_0 = \left\lfloor \frac{\log\left(\frac{1}{1-\sqrt{\alpha}\sigma_{H_0}}\right)}{\log(1+\alpha\sqrt{\chi\epsilon})} \right\rfloor, \text{ where } \sigma_{H_0} = \frac{2p\sqrt{p}hR_{\max}(hG^2+L)}{1-\gamma},$$

after at most $j \leq \hat{\kappa}_0$ steps, we have

$$\mathbb{E}[J(\theta_{k+j})] - J(\theta_k) \ge \alpha^2 \iota^2 \sqrt{\chi \epsilon},$$

where ι is a positive constant defined in (132).

Notations There are many notations in this section for convenience of reference, we list key notations and constants in the following table.

$$\begin{array}{lll} H_0 &: & H_0 =: \nabla^2 J(\theta_0). \\ \{\lambda_i\}_{i=1}^p &: & \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p \text{ are the eigenvalues of the matrix } \nabla^2 J(\theta_0). \\ \Lambda &: & \text{The operator norm of matrix } H_0, \text{ i.e., } \Lambda = \max_{1 \leq i \leq p} \{|\lambda_i|\}. \\ \widehat{J}(\theta) &: & \text{The second order approximation of } J(\theta), \text{ and it is defined in (58).} \\ \widehat{H}_0 &: & \text{An estimator of } H_0, \text{ it is defined in (59)} \\ \widehat{\theta}_k &: & \text{The iteration defined in (60) that is generated according to problem } \\ \max_{\alpha \in \mathcal{J}} \widehat{J}(\theta). \\ \xi_k &: & \xi_k = g(\theta_k) - \nabla J(\theta_k) \text{ that is defined in (32).} \\ \widehat{\xi}_k &: & \widehat{\xi}_k = (\widehat{H}_0 - H_0)(\widehat{\theta}_k - \widehat{\theta}_0). \\ C_1 &: & C_1 \text{ is defined in (86).} \\ C_2, C_3 &: & C_2, C_3 \text{ are defined in (65).} \\ \widehat{\kappa}_0 &: & \text{A positive integer defined in (84).} \\ \Delta H_k &: & \Delta H_k = \nabla^2 J(\theta_k) - H_0 = \nabla^2 J(\theta_k) - \nabla^2 J(\theta_0). \\ \Delta_k &: & \Delta_k = \nabla J(\theta_k) - \nabla \widehat{J}(\widehat{\theta}_k). \\ \widehat{\phi}, \phi &: & \widehat{\phi} = \widehat{\theta}_{k+1} - \theta_0 = \widehat{\theta}_{k+1} - \widehat{\theta}_0 \text{ and } \phi = \theta_{k+1} - \widehat{\theta}_{k+1}. \\ \left\| \frac{d_{\pi \theta}^{\pi \theta}}{\theta_0} \right\|_{\infty} &= \sup_{s \in S} \frac{d_{\pi \theta}^{\pi \theta}(s)}{\theta_0(s)}. \\ \sigma_{H_0} &: & \text{It is defined in (78).} \\ B_k, \mathcal{C}_k &: & \text{Two events defined in (101) and (102).} \\ B_1 - B_4 &: & \text{They are defined in (109).} \\ E_1, E_2 &: & \text{They are defined in (115).} \\ \beta &: & \beta = \frac{2\sqrt{\alpha \log \frac{1}{\alpha} D_1 B_1 + D_2}}{\Lambda^2 + 2\Lambda} \end{array}$$

Without loss of generality, in the proof, we consider the initial θ_0 falls into the region \mathcal{L}_2 .

Organization and Key Ideas in This Section It is very technical to achieve the result of Proposition 2, we outline some necessary intermediate results in the following Section E.1. Concretely, Lemma 10 and Lemma 12 play a key role in the proof of Proposition 2. We utilize the second order information of the expected return $J(\theta)$ as following two key steps:

• Firstly, let $\widehat{J}(\theta)$ (defined in (58)) be the second order approximation of $J(\theta)$, we consider the following optimization problem

$$\max_{\theta} \widehat{J}(\theta), \tag{57}$$

and construct a sequence $\{\widehat{\theta}_k\}_{k\geq 0}$ (defined in the below Eq.(60)) to solve the problem (57). Lemma 10 illustrates if an initial iteration $\widehat{\theta}_0(=\theta_0)$ falls into the region

$$\mathcal{L}_2 = \{\theta \in \mathbb{R}^p : \|\nabla J(\theta)\|_2 \le \epsilon\} \cap \{\theta \in \mathbb{R}^p : \lambda_{\max}(\nabla^2 J(\theta)) \ge \sqrt{\chi \epsilon}\},\$$

then the following facts happen: $\hat{\theta}_{k+1}$ closes to the initial point $\hat{\theta}_0$ for a long time and $\|\nabla \hat{J}(\hat{\theta}_{k+1})\|_2$ can be small for a long time. Lemma 7-9 provide some intermediate results for the proof of Lemma 10, we will provide all the details of them step by step in the following Section E.2 to E.4. The proof of Lemma 10 lies in Section E.5.

• Then, under above conditions, we provide Lemma 12, which illustrates if an initial iteration $\hat{\theta}_0$ falls into the region \mathcal{L}_2 , for a proper step-size, the iteration generated according to the problem (57) can be closed to the solution of policy optimization (2) with high probability. We present the precise result in (71). Similarly, the policy gradient estimator of the problem (2) can be closed to the policy distribution of (57), which is presented in (70).

E.1 Summary of Preliminary Results

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$ be the eigenvalues of the matrix $\nabla^2 J(\theta_0)$. We use H_0 to denote the matrix $\nabla^2 J(\theta_0)$, in this section, we assume the initial point θ_0 satisfies

$$\lambda_{\max}(\nabla^2 J(\theta_0)) = \lambda_{\max}(H_0) = \lambda_p \ge \sqrt{\chi\epsilon}, \qquad \|\nabla J(\theta_0)\|_2 \le \epsilon.$$

Furthermore, by the definition of the operator norm of H_0 , we have

$$||H_0||_{op} = \max_{1 \le i \le p} \{|\lambda_i|\} = \max\{|\lambda_1|, \lambda_p\} =: \Lambda \ge \sqrt{\chi\epsilon}.$$

We use $\widehat{J}(\theta)$ to denote the second order approximation of $J(\theta)$, i.e., let $H_0 = \nabla^2 J(\theta_0) \in \mathbb{R}^{p \times p}$,

$$\widehat{J}(\theta) = J(\theta_0) + \nabla J(\theta_0)^\top (\theta - \theta_0) + \frac{1}{2} (\theta - \theta_0)^\top H_0(\theta - \theta_0).$$
(58)

Let $\hat{\tau}_0 = {\{\hat{s}_t^0, \hat{a}_t^0, \hat{r}_{t+1}^0\}_{t=0}^h \sim \pi_{\hat{\theta}_0}}$, and we introduce a notation $\hat{\Phi}(\theta)$ as follows,

$$\widehat{\Phi}(\theta) = \sum_{t=0}^{h} \sum_{i=t}^{h} \gamma^{i} \widehat{r}_{i+1}^{0}(\widehat{s}_{i}^{0}, \widehat{a}_{i}^{0}) \log \pi_{\theta}(\widehat{s}_{h}^{0}, \widehat{a}_{h}^{0}).$$

According to the section 7.2 of (Shen et al., 2019), we conduct an unbiased estimator of H_0 as:

$$\widehat{H}_0 = \nabla \widehat{\Phi}(\theta) \nabla \log p(\widehat{\tau}_0; \pi_\theta)^\top + \nabla^2 \widehat{\Phi}(\theta) \Big|_{\theta = \widehat{\theta}_0},$$
(59)

where $p(\hat{\tau}_0; \pi_{\hat{\theta}_0}) = \rho_0(\hat{s}_0^0) \prod_{t=0}^h P(\hat{s}_{t+1}^0 | \hat{s}_t^0, \hat{a}_t^0) \pi_{\hat{\theta}_0}(\hat{a}_t^0 | \hat{s}_t^0)$ is the probability of generating $\hat{\tau}_0$ according to the policy $\pi_{\hat{\theta}_0}$.

Lemma 7 (Boundedness of $\|\widehat{H}_0 - H_0\|_{op}$). Let $H_0 = \nabla^2 J(\theta_0)$, \widehat{H}_0 defined in (59) is an estimator of H_0 . Under Assumption 1-3, the following holds

$$\|\widehat{H}_0 - H_0\|_{op} \le 2\frac{p\sqrt{p}hR_{\max}}{1-\gamma} \left(hG^2 + L\right) =: \sigma_{H_0}.$$

Now, we consider a coupled sequence $\{\widehat{\theta}_k\}_{k\geq 0}$ is the iteration of policy gradient solution on the second order approximation function $\widehat{J}(\theta)$ defined in (58), i.e., $\{\widehat{\theta}_k\}_{k\geq 0}$ solves the problem $\max_{\theta} \widehat{J}(\theta)$ along the direction of a policy gradient estimator of $\nabla \widehat{J}(\theta)$. Concretely, we set the initial value of $\widehat{\theta}_0$ as the same consideration with policy optimization problem (2), i.e., $\widehat{\theta}_0 = \theta_0$. Then define the update rule of $\widehat{\theta}_k$ as follows,

$$\widehat{\theta}_0 = \theta_0, \qquad \widehat{\theta}_{k+1} = \widehat{\theta}_k + \alpha \widehat{\nabla \widehat{J}(\widehat{\theta}_k)}, \qquad (60)$$

where $\widehat{\nabla \hat{J}(\cdot)}$ is an estimator of the gradient function $\nabla \hat{J}(\cdot)$, and α is step-size. Taking the gradient of the function $\hat{J}(\theta)$ (58), we have

$$\nabla \widehat{J}(\theta) = \nabla J(\theta_0) + H_0(\theta - \theta_0), \tag{61}$$

for the iteration (60), we define an estimator of $\nabla \widehat{J}(\widehat{\theta}_k)$ as follows,

$$\widehat{\nabla J}(\widehat{\widehat{\theta}_k}) = g(\widehat{\theta}_0) + \widehat{H}_0(\widehat{\theta}_k - \widehat{\theta}_0), \tag{62}$$

where \widehat{H}_0 is defined in (59). Recall ξ_k defined in (32), since $\widehat{\theta}_0 = \theta_0$, then $\xi_0 = g(\widehat{\theta}_0) - \nabla J(\widehat{\theta}_0)$. Let $\widehat{\xi}_k = (\widehat{H}_0 - H_0)(\widehat{\theta}_k - \widehat{\theta}_0)$, since $\nabla \widehat{J}(\widehat{\theta}_k) = \nabla J(\widehat{\theta}_0) + H_0(\widehat{\theta}_k - \widehat{\theta}_0)$, we can rewrite (60):

$$\widehat{\theta}_{k+1} \underbrace{\underbrace{(60),(62)}}_{=\widehat{\theta}_k} \widehat{\theta}_k + \alpha \Big(g(\widehat{\theta}_0) + \widehat{H}_0(\widehat{\theta}_k - \widehat{\theta}_0) \Big)$$

$$= \widehat{\theta}_k + \alpha \Big(\underbrace{g(\widehat{\theta}_0) - \nabla J(\widehat{\theta}_0)}_{=\xi_0} + \underbrace{\nabla J(\widehat{\theta}_0) + H_0(\widehat{\theta}_k - \widehat{\theta}_0)}_{=\nabla \widehat{J}(\widehat{\theta}_k)} + \underbrace{(\widehat{H}_0 - H_0)(\widehat{\theta}_k - \widehat{\theta}_0)}_{=\widehat{\xi}_k} \Big)$$

$$= \widehat{\theta}_k + \alpha \Big(\nabla \widehat{J}(\widehat{\theta}_k) + \widehat{\xi}_k + \xi_0 \Big).$$
(63)

Staring from (64), after some careful calculations, we have

$$\widehat{\theta}_{k+1} - \widehat{\theta}_0 = \alpha \underbrace{\sum_{j=0}^k (I + \alpha H_0)^j \nabla J(\widehat{\theta}_0)}_{-:I_0} + \alpha \underbrace{\sum_{j=0}^k (I + \alpha H_0)^{k-j} \left(\widehat{\xi}_j + \xi_0\right)}_{-:I_0}, \tag{65}$$

$$\nabla \widehat{J}(\widehat{\theta}_{k+1}) = (I + \alpha H_0)^k \nabla J(\theta_0) + \alpha \underbrace{\sum_{j=0}^k (I + \alpha H_0)^{k-j} (\widehat{\xi}_j + \xi_0)}_{=:I_1}.$$
(66)

Lemma 8 (Boundedness of Term I_1 with High Probability). Under the conditions of Assumption 1-3 and Proposition 2, consider the term $I_1 = \sum_{j=0}^k (I + \alpha H_0)^{k-j} (\hat{\xi}_j + \xi_0)$ defined in (65)-(66). Then, there exists a positive integer $\hat{\kappa}_0$, such that for each $k \in [0, \hat{\kappa}_0]$, for any $\delta \in (0, 1)$, we have

$$\mathbb{P}\left(\left\|\sum_{j=0}^{k} (I+\alpha H_0)^{k-j} \left(\widehat{\xi}_j + \xi_0\right)\right\|_2 \le 2C_1 \sqrt{\widehat{\kappa}_0 \log \frac{4}{\delta}}\right) \ge 1-\delta,$$

where C_1 is defined in (86).

Lemma 9 (Boundedness of Term I_2 with High Probability). Under the conditions of Assumption 1-3 and Proposition 2, consider the term $I_2 = \sum_{j=0}^{k} (I + \alpha H_0)^j \nabla J(\hat{\theta}_0)$ defined in (65). Then, there exists a positive integer $\hat{\kappa}_0$, such that for each $k \in [0, \hat{\kappa}_0]$, we have

$$\left\|I_2\right\|_2 = \left\|\sum_{j=0}^k (I + \alpha H_0)^j \nabla J(\widehat{\theta}_0)\right\|_2 \le \frac{\epsilon}{1 - \sqrt{\alpha} \sigma_{H_0}}.$$

From the results of Lemma 8 and Lemma 9, combine the result of (65)-(66), we have the boundedness of $\|\widehat{\theta}_{k+1} - \widehat{\theta}_0\|_2$ and $\|\nabla \widehat{J}(\widehat{\theta}_{k+1})\|_2$ as following Lemma 10.

Lemma 10. Under Assumption 1-3, consider the sequence $\{\theta_k\}_{k\geq 0}$ generated by (5), $\{\widehat{\theta}_k\}_{k\geq 0}$ generated by (60), and the initial point satisfies $\theta_0 = \widehat{\theta}_0$. Let $H_0 = \nabla^2 J(\theta_0)$, the initial point θ_0 also satisfies $\lambda_{\max}(H_0) \geq \sqrt{\chi\epsilon}$, $\|\nabla J(\theta_0)\|_2 \leq \epsilon$. Let $\widehat{J}(\theta)$ (58) be the second order approximation of the expected return $J(\theta)$. Let

$$\sigma_{H_0} = \frac{2p\sqrt{p}hR_{\max}(hG^2 + L)}{1 - \gamma}, \alpha < \min\left\{\frac{1}{\sigma_{H_0}}, \frac{1}{\sigma_{H_0}^2}\right\}, \left\lfloor\frac{\log\left(\frac{1}{1 - \sqrt{\alpha}\sigma_{H_0}}\right)}{\log(1 + \alpha\sqrt{\chi\epsilon})}\right\rfloor =: \hat{\kappa}_0.$$

For each $k \in [0, \hat{\kappa}_0]$, for any $\delta \in (0, 1)$, we have

$$\mathbb{P}\left(\left\|\widehat{\theta}_{k+1} - \widehat{\theta}_{0}\right\|_{2} \leq \alpha \left(2C_{1}\sqrt{\widehat{\kappa}_{0}\log\frac{4}{\delta}} + \frac{\epsilon}{1 - \sqrt{\alpha}\sigma_{H_{0}}}\right)\right) \geq 1 - \delta.$$
$$\mathbb{P}\left(\left\|\nabla\widehat{J}(\widehat{\theta}_{k+1})\right\|_{2} \leq 2\alpha C_{1}\sqrt{\widehat{\kappa}_{0}\log\frac{4}{\delta}} + \frac{\epsilon}{1 - \sqrt{\alpha}\sigma_{H_{0}}}\right) \geq 1 - \delta.$$
where $C_{1} = \frac{1}{1 - \sqrt{\alpha}\sigma_{H_{0}}} \left(\frac{\sigma_{H_{0}}GR_{\max}}{1 - \gamma} \left(\frac{1}{\sqrt{\chi\epsilon}(1 - \sqrt{\alpha}\sigma_{H_{0}})} + \frac{\alpha}{1 - \alpha\sigma_{H_{0}}}\right) + \sigma\right).$

Now, we turn to consider the term $\nabla J(\theta_k) - \nabla \widehat{J}(\widehat{\theta}_k) =: \Delta_k$ and $\theta_{k+1} - \widehat{\theta}_{k+1}$. We give the partition of Δ_k and $\theta_{k+1} - \widehat{\theta}_{k+1}$ as below.

Let's calculate the policy gradient $\nabla J(\theta_{k+1})$ as follows,

$$\nabla J(\theta_{k+1}) = \nabla J(\theta_k) + \nabla^2 J(\theta_k)(\theta_{k+1} - \theta_k) + e_k$$

= $(1 + \alpha H_0)\nabla J(\theta_k) + \alpha H_0(\hat{\xi}_k + \xi_0) + \alpha \Delta H_k(\nabla J(\theta_k) + \hat{\xi}_k + \xi_0) + e_k,$ (67)

where

$$e_{k} = \int_{0}^{1} \left[\nabla^{2} J(\theta_{k} + x(\theta_{k+1} - \theta_{k})) - \nabla^{2} J(\theta_{k}) \right] \mathrm{d}(\theta_{k+1} - \theta_{k}) x,$$
$$\Delta H_{k} = \nabla^{2} J(\theta_{k}) - H_{0} = \nabla^{2} J(\theta_{k}) - \nabla^{2} J(\theta_{0}).$$

Furthermore, let $\Delta_k = \nabla J(\theta_k) - \nabla J(\theta_k)$, then according to (66) and (67), after some careful calculations, we have the following partition of the term $\nabla J(\theta_{k+1}) - \nabla \widehat{J}(\widehat{\theta}_{k+1})$ and $\theta_{k+1} - \widehat{\theta}_{k+1}$

$$\Delta_{k+1} = \nabla J(\theta_{k+1}) - \nabla J(\theta_{k+1}) = (I + \alpha H_0) \Delta_k + \alpha \Delta H_k (\Delta_k + \nabla \widehat{J}(\widehat{\theta}_k)) + \alpha \Delta H_k (\widehat{\xi}_k + \xi_0) + e_k$$
(68)

$$\theta_{k+1} = \widehat{\theta}_{k+1} + \alpha \sum_{j=0}^{k} \Delta_j \tag{69}$$

Finally, we need the boundedness of $\|\nabla J(\theta_k)\|_2$ to prove the Lemma 12, we present it below.

Lemma 11. The term $\|\nabla J(\theta_k)\|_2$ is upper-bounded as follows,

$$\begin{aligned} \|\nabla J(\theta_k)\|_2 \stackrel{(\mathbf{a})}{=} \|\nabla J(\theta_\star) - \nabla J(\theta_k)\|_2 &\leq \left\|\frac{d_{\rho_0}^{\pi_{\theta_\star}}}{\rho_0}\right\|_{\infty} \frac{(1+\gamma|\mathcal{S}|)|\mathcal{A}|^2|\mathcal{S}|R_{\max}}{(1-\gamma)^2}, \end{aligned}$$
where $\left\|\frac{d_{\rho_0}^{\pi_{\theta_\star}}}{\rho_0}\right\|_{\infty} = \max_{s\in\mathcal{S}} \frac{d_{\rho_0}^{\pi_{\theta_\star}}(s)}{\rho_0(s)}, \ Eq.(\mathbf{a}) \ holds \ since \ \nabla J(\theta_\star) = 0. \end{aligned}$

Lemma 12. Under Assumption 1-3, consider the sequence $\{\theta_k\}_{k\geq 0}$ generated by (5), $\{\widehat{\theta}_k\}_{k\geq 0}$ generated by (60), and the initial point satisfies $\theta_0 = \widehat{\theta}_0$. Let $H_0 = \nabla^2 J(\theta_0)$, the initial point θ_0 also satisfies $\lambda_{\max}(H_0) \geq \sqrt{\chi \epsilon}$, $\|\nabla J(\theta_0)\|_2 \leq \epsilon$. Let $\widehat{J}(\theta)$ (58) be the second order approximation of the expected return $J(\theta)$. Let $\sigma_{H_0} = \frac{2p\sqrt{p}hR_{\max}(hG^2+L)}{1-\gamma}$, $\alpha < \frac{1}{\sigma_{H_0}}$, $\left|\frac{\log\left(\frac{1}{1-\sqrt{\alpha}\sigma_{H_0}}\right)}{\log(1+\alpha\sqrt{\chi \epsilon})}\right| + 1 =: \widehat{\kappa}_0$. For each

 $k \in [0, \widehat{\kappa}_0]$, the following holds,

$$\mathbb{P}\left(\max\left\{\|\Delta_k\|_2, \|\Delta_k\|_2^2\right\} \ge C_4 \sqrt{\alpha^3 \log \frac{1}{\alpha}}\right) \le \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}} + o(\alpha^{\frac{3}{2}}),\tag{70}$$

$$\mathbb{P}\left(\left\|\theta_{k+1} - \widehat{\theta}_{k+1}\right\|_{2} \le \alpha^{2} \sqrt{\log \frac{1}{\alpha}} \frac{\sigma_{H_{0}}}{\sqrt{\chi\epsilon}} C_{4} + o\left(\alpha^{2} \sqrt{\log \frac{1}{\alpha}}\right)\right) \le \frac{\sigma_{H_{0}}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}} + o(\alpha^{\frac{3}{2}}).$$
(71)

E.2 Proof of Lemma 7

Proof. (of Lemma7). We need two results of operator norms to show the boundedness of $\|\hat{H}_0 - H_0\|_{op}$. For any matrix $A = (a_{i,j}) \in \mathbb{R}^{p \times p}, B \in \mathbb{R}^{p \times p}$, the following holds,

$$||A + B||_{op} \le ||A||_{op} + ||B||_{op}, \tag{72}$$

$$\max_{i,j}\{|a_{i,j}|\} \le \|A\|_{op} \le p\sqrt{p} \max_{i,j}\{|a_{i,j}|\}.$$
(73)

For the proof of (72) and (73), please refer to Theorem 3.4 of a lecture provided in

https://kconrad.math.uconn.edu/blurbs/linmultialg/matrixnorm.pdf.

Using the result of (73), we have

$$\|\widehat{H}_0 - H_0\|_{op} \le \|\widehat{H}_0\|_{op} + \|H_0\|_{op},$$

to achieve the boundedness of $\|\hat{H}_0 - H_0\|_{op}$, we need to bound $\|\hat{H}_0\|_{op}$, $\|H_0\|_{op}$ correspondingly.

Recall $\tau_0 = \{s_t^0, a_t^0, r_{t+1}^0\}_{t=0}^h \sim \pi_{\theta_0}$, let $\Phi(\theta) = \sum_{t=0}^h \sum_{i=t}^h \gamma^i r_{i+1}^0(s_i^0, a_i^0) \log \pi_{\theta}(s_h^0, a_h^0)$. According to section 7.2 of (Shen et al., 2019), the second order derivative of $J(\theta)$ is

$$\nabla^2 J(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_0}} \Big[\nabla \Phi(\theta) \nabla \log p(\tau_0; \pi_\theta)^\top + \nabla^2 \Phi(\theta), \Big]$$
(74)

where $p(\tau_0; \pi_{\theta_0}) = \rho_0(s_0^0) \prod_{t=0}^h P(s_{t+1}^0 | s_t^0, a_t^0) \pi_{\theta_0}(a_t^0 | s_t^0)$ is the probability of generating τ_0 according to the policy π_{θ_0} . From the result of (74), if we get the boundedness of $\nabla \Phi(\theta) \nabla \log p(\tau_0; \pi_{\theta})^\top + \nabla^2 \Phi(\theta)$, then the boundedness of $||H_0||_{op} = ||\nabla^2 J(\theta_0)||_{op}$ is clear.

Now, we will bound the operator norm of the matrix $\nabla \Phi(\theta) \nabla \log p(\tau_0; \pi_\theta)^\top + \nabla^2 \Phi(\theta)$. We need a simple fact: if $a = (a_1, a_2, \cdots, a_p)^\top$, $b = (b_1, b_2, \cdots, b_p) \in \mathbb{R}^p$, and $A = ab^\top$, then we have

$$\|A\|_{op} = \|ab^{\top}\|_{op} \stackrel{(73)}{\leq} p\sqrt{p} \max_{i,j} |a_i b_j| \leq p\sqrt{p} \max_i |a_i| \cdot \max_j |b_j|.$$
(75)

The results of (73),(75) imply that to achieve the boundedness of the operator norm of the matrix $\nabla \Phi(\theta) \nabla \log p(\tau_0; \pi_{\theta})^\top + \nabla^2 \Phi(\theta)$, we need to bound the elements of the vectors: $\nabla \Phi(\theta), \nabla \log p(\tau_0; \pi_{\theta})^\top$, and bound each element of the Hessian matrix $\nabla^2 \Phi(\theta)$.

Recall Assumption 1, for each $1 \leq j \leq p$, we have

$$\left| [\nabla \Phi(\theta)]_j \right| = \left| \left[\sum_{t=0}^h \sum_{i=t}^h \gamma^i r_{i+1}^0(s_i^0, a_i^0) \underbrace{\nabla \log \pi_\theta(s_h^0, a_h^0)}_{\leq G} \right]_j \right| \le \frac{hGR_{\max}}{1 - \gamma},$$

$$\left| [\nabla \log p(\tau_0; \pi_\theta)]_j \right| = \left| \left[\sum_{t=0}^h \rho_0(s_0^0) P(s_{t+1}^0 | s_t^0, a_t^0) \underbrace{\nabla \log \pi_\theta(s_h^0, a_h^0)}_{\leq G} \right]_j \right| \le hG,$$

where $[\cdot]_j$ denotes the *j*-the coordinate component of a vector. Combining above two results, we have

$$\|\nabla\Phi(\theta)\nabla\log p(\tau_0;\pi_\theta)^{\top}\|_{op} \stackrel{(75)}{\leq} p\sqrt{p}\frac{hGR_{\max}}{1-\gamma} \cdot hG = p\sqrt{p}\frac{h^2G^2R_{\max}}{1-\gamma}.$$
(76)

Now, we bound on the term $\|\nabla^2 \Phi(\theta)\|_{op}$. Let $[\cdot]_{i,j}$ denotes the (i, j)-the coordinate component of a matrix, then for each element of $\|\nabla^2 \Phi(\theta)\|_{op}$, we have

$$\left| [\nabla^2 \Phi(\theta)]_{i,j} \right| = \left| \sum_{t=0}^h \sum_{i=t}^h \gamma^i r_{i+1}^0(s_i^0, a_i^0) \underbrace{\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log \pi_\theta(s_h^0, a_h^0))}_{\substack{(7) \\ \leq F}} \right| \le \frac{hLR_{\max}}{1 - \gamma}$$

which implies

$$\|\nabla^2 \Phi(\theta)\|_{op} \stackrel{(73)}{\leq} p_{\sqrt{p}} \frac{hLR_{\max}}{1-\gamma}.$$
(77)

By the results of (76) and (77), we have

$$\begin{aligned} \|H_0\|_{op} &= \|\nabla J(\theta_0)\|_{op} \le \|\nabla^2 \Phi(\theta) \nabla \log p(\tau_0; \pi_\theta)^\top\|_{op} + \|\nabla^2 \Phi(\theta)\|_{op} \\ &\le \frac{p\sqrt{p}hR_{\max}}{1-\gamma} \Big(hG^2 + L\Big). \end{aligned}$$

As the same analysis with $||H_0||_{op}$, we have $||\widehat{H}_0||_{op} \leq \frac{p\sqrt{p}hR_{\max}}{1-\gamma} (hG^2 + L)$. Thus, we have

$$\|\widehat{H}_0 - H_0\|_{op} \le \|\widehat{H}_0\|_{op} + \|H_0\|_{op} \le 2\frac{p\sqrt{p}hR_{\max}}{1-\gamma} \left(hG^2 + L\right) =: \sigma_{H_0}.$$
(78)

This concludes the proof.

E.3 Proof of Lemma 8

Proof. (of Lemma 8) Recall $I_1 = \sum_{j=0}^k (I + \alpha H_0)^{k-j} (\hat{\xi}_j + \xi_0)$, which implies to bound I_1 , we need to bound $\hat{\xi}_k$.

Firstly, we decompose the term $\hat{\xi}_k = (\hat{H}_0 - H_0)(\hat{\theta}_k - \hat{\theta}_0)$. Consider the update (63), we have

$$\widehat{\theta}_{k+1} - \widehat{\theta}_k = (I + \alpha \widehat{H}_0)(\widehat{\theta}_k - \widehat{\theta}_{k-1})$$

$$= (I + \alpha H_0)(\widehat{\theta}_k - \widehat{\theta}_{k-1}) + \alpha (\widehat{H}_0 - H_0)(\widehat{\theta}_k - \widehat{\theta}_{k-1})$$

$$= \cdots \cdots$$

$$= (I + \alpha H_0)^k (\widehat{\theta}_1 - \widehat{\theta}_0) + \alpha^k (\widehat{H}_0 - H_0)^k (\widehat{\theta}_1 - \widehat{\theta}_0).$$
(80)

Recall $\hat{\xi}_k = (\hat{H}_0 - H_0)(\hat{\theta}_k - \hat{\theta}_0)$, according to the result of (80), we rewrite $\hat{\xi}_k$ as follows,

$$\widehat{\xi}_k = (\widehat{H}_0 - H_0) \left(\sum_{j=0}^{k-1} \left((I + \alpha H_0)^j (\widehat{\theta}_1 - \widehat{\theta}_0) + \alpha^j (\widehat{H}_0 - H_0)^j (\widehat{\theta}_1 - \widehat{\theta}_0) \right) \right).$$
(81)

Boundedness of $\hat{\xi}_k = (\hat{H}_0 - H_0)(\hat{\theta}_k - \hat{\theta}_0).$

$$\begin{aligned} \|\widehat{\xi}_{k}\|_{2} \stackrel{(\$1)}{=} \|(\widehat{H}_{0} - H_{0}) \left(\sum_{j=0}^{k-1} \left((I + \alpha H_{0})^{j} (\widehat{\theta}_{1} - \widehat{\theta}_{0}) + \alpha^{j} (\widehat{H}_{0} - H_{0})^{j} (\widehat{\theta}_{1} - \widehat{\theta}_{0}) \right) \right) \|_{2} \\ \leq \|(\widehat{H}_{0} - H_{0})\|_{op} \underbrace{\|\widehat{\theta}_{1} - \widehat{\theta}_{0}\|_{2}}_{=\alpha \|g(\theta_{0})\|_{2}} \left(\sum_{j=0}^{k-1} \left((1 + \alpha \Lambda)^{j} + \alpha^{j} \|\widehat{H}_{0} - H_{0}\|_{op}^{j} \right) \right) \\ \stackrel{(78)}{\leq} \alpha \sigma_{H_{0}} \|g(\theta_{0})\|_{2} \left(\sum_{j=0}^{k-1} \left((1 + \alpha \Lambda)^{j} + \alpha^{j} \sigma_{H_{0}}^{j} \right) \right) \\ \leq \alpha \sigma_{H_{0}} \|g(\theta_{0})\|_{2} \left(\frac{(1 + \alpha \Lambda)^{k-1}}{\alpha \Lambda} + \frac{1 - \alpha^{k-1} \sigma_{H_{0}}^{k-1}}{1 - \alpha \sigma_{H_{0}}} \right). \end{aligned}$$
(82)

Let

$$(1 + \alpha \Lambda)^{k-1} \le \frac{1}{1 - \sqrt{\alpha} \sigma_{H_0}},\tag{83}$$

which requires $k \leq \left\lfloor \frac{\log\left(\frac{1}{1-\sqrt{\alpha\sigma_{H_0}}}\right)}{\log(1+\alpha\Lambda)} \right\rfloor$. For a enough small α , the term $\left\lfloor \frac{\log\left(\frac{1}{1-\sqrt{\alpha\sigma_{H_0}}}\right)}{\log(1+\alpha\Lambda)} \right\rfloor$ decreases as Λ increases. Recall $\Lambda \geq \sqrt{\chi\epsilon}$, then we have

$$k \le \left\lfloor \frac{\log\left(\frac{1}{1 - \sqrt{\alpha}\sigma_{H_0}}\right)}{\log(1 + \alpha\Lambda)} \right\rfloor \le \left\lfloor \frac{\log\left(\frac{1}{1 - \sqrt{\alpha}\sigma_{H_0}}\right)}{\log(1 + \alpha\sqrt{\chi\epsilon})} \right\rfloor =: \widehat{\kappa}_0.$$
(84)

Thus, for all $k \leq \hat{\kappa}_0$, from (82), and $\Lambda \geq \sqrt{\chi \epsilon}$, we have

$$\begin{aligned} \|\widehat{\xi}_{k}\|_{2} &\leq \alpha \sigma_{H_{0}} \|g(\theta_{0})\|_{2} \left(\frac{1}{\alpha \Lambda (1-\sqrt{\alpha}\sigma_{H_{0}})} + \frac{1}{1-\alpha\sigma_{H_{0}}}\right) \\ &\leq \alpha \sigma_{H_{0}} \|g(\theta_{0})\|_{2} \left(\frac{1}{\alpha \sqrt{\chi \epsilon}(1-\sqrt{\alpha}\sigma_{H_{0}})} + \frac{1}{1-\alpha\sigma_{H_{0}}}\right) \\ &\stackrel{(44)}{\leq} \frac{\sigma_{H_{0}} GR_{\max}}{1-\gamma} \left(\frac{1}{\sqrt{\chi \epsilon}(1-\sqrt{\alpha}\sigma_{H_{0}})} + \frac{\alpha}{1-\alpha\sigma_{H_{0}}}\right). \end{aligned}$$
(85)

It is noteworthy that to ensure the term $1 - \alpha \sigma_{H_0}$ and $1 - \sqrt{\alpha} \sigma_{H_0}$ keep positive, we require $\alpha < \min\{\frac{1}{\sigma_{H_0}}, \frac{1}{\sigma_{H_0}^2}\}$.

Boundedness of I_1 in Eq.(65)/(66). Furthermore, for each $0 \le j \le \hat{\kappa}_0$, we have

$$\|(I + \alpha H_0)^{k-j} (\hat{\xi}_j + \xi_0)\|_2 \le \|I + \alpha H_0\|_{op}^{\hat{\kappa}_0} \|\hat{\xi}_j + \xi_0\|_2 = (1 + \alpha \Lambda)^{\hat{\kappa}_0} \|\hat{\xi}_j + \xi_0\|_2 \le (1 + \alpha \Lambda)^{\hat{\kappa}_0} \left(\|\hat{\xi}_j\|_2 + \|\xi_0\|_2\right) \stackrel{(85)}{\le} \frac{1}{1 - \sqrt{\alpha}\sigma_{H_0}} \left(\frac{\sigma_{H_0} GR_{\max}}{1 - \gamma} \left(\frac{1}{\sqrt{\chi\epsilon}(1 - \sqrt{\alpha}\sigma_{H_0})} + \frac{\alpha}{1 - \alpha\sigma_{H_0}}\right) + \sigma\right) =: C_1.$$
(86)

Finally, for each $0 \leq j \leq \widehat{\kappa}_0$ and $k \leq \widehat{\kappa}_0$ we have

$$\mathbb{E}\Big[\left(I + \alpha H_0\right)^{k-j} \left(\widehat{\xi}_j + \xi_0\right)\Big] = 0.$$
(87)

Recall Hoeffding inequality, and the result of (86),(87), for each $k \in [0, \hat{\kappa}_0]$, for any $\delta \in (0, 1)$,

$$\mathbb{P}\left(\left\|\sum_{j=0}^{k} (I+\alpha H_0)^{k-j} \left(\widehat{\xi}_j + \xi_0\right)\right\|_2 \le 2C_1 \sqrt{\widehat{\kappa}_0 \log \frac{4}{\delta}}\right) \ge 1 - \delta.$$
(88)

Remark 3. For a small α , the term C_1 (86) can be rewritten as follows,

$$C_1 = \frac{\sigma_{H_0} G R_{\max}}{(1 - \gamma) \sqrt{\chi \epsilon}} + \sigma + \mathcal{O}(\alpha).$$
(89)

E.4 Proof of Lemma 9

Proof. (of Lemma 9). Recall $\hat{\theta}_0 = \theta_0$, then $\|\nabla J(\hat{\theta}_0)\| \leq \epsilon$. Thus, for each $k \in [0, \hat{\kappa}_0]$, we have

$$\left\|I_{2}\right\|_{2} = \left\|\sum_{j=0}^{k} (I + \alpha H_{0})^{j} \nabla J(\widehat{\theta}_{0})\right\|_{2} \leq \sum_{j=0}^{k} (1 + \alpha \Lambda)^{j} \left\|\nabla J(\widehat{\theta}_{0})\right\|_{2} \leq \frac{\epsilon}{1 - \sqrt{\alpha}\sigma_{H_{0}}}.$$
 (90)

This concludes the proof.

E.5 Proof of Lemma 10

Proof. (of Lemma 10)

Boundedness for the term $\hat{\theta}_{k+1} - \hat{\theta}_0$ defined in Eq.(65) with high probability.

Recall the term $\hat{\theta}_{k+1} - \hat{\theta}_0 = \alpha I_1 + \alpha I_2$ defined in (65), and results of (88) and (90) show the boundedness of I_1 and I_2 correspondingly. Then for each $k \in [0, \hat{\kappa}_0]$, for any $\delta \in (0, 1)$, we have

$$\mathbb{P}\left(\left\|\widehat{\theta}_{k+1} - \widehat{\theta}_0\right\|_2 \le \alpha \left(2C_1 \sqrt{\widehat{\kappa}_0 \log \frac{4}{\delta}} + \frac{\epsilon}{1 - \sqrt{\alpha}\sigma_{H_0}}\right)\right) \ge 1 - \delta.$$
(91)

Boundedness for the term $\nabla \widehat{J}(\widehat{\theta}_{k+1})$ defined in Eq.(66) with high probability.

Firstly, we bound the term $(I + \alpha H_0)^k \nabla J(\theta_0)$ in Eq.(66). Recall the condition of $\hat{\kappa}_0$ defined in (84), the result of (99), and the initial condition of $\|\nabla J(\theta_0)\|_2 \leq \epsilon$, then we have

$$\left\| (I + \alpha H_0)^k \nabla J(\theta_0) \right\|_2 \le \left\| I + \alpha H_0 \right\|_{op}^k \| \nabla J(\theta_0) \|_2 = (1 + \alpha \Lambda)^k \| \nabla J(\theta_0) \|_2 \le \frac{\epsilon}{1 - \sqrt{\alpha} \sigma_{H_0}}.$$

Furthermore, from the result of (88), for each $k \in [0, \hat{\kappa}_0]$, the following holds with probability at least $1 - \delta$, $\|\nabla \widehat{J}(\widehat{\theta}_{k+1})\|_2 = \left\| (I + \alpha H_0)^k \nabla J(\theta_0) + \alpha \sum_{j=0}^k (I + \alpha H_0)^{k-j} (\widehat{\xi}_j + \xi_0) \right\|_2 \le \alpha 2C_1 \sqrt{\widehat{\kappa}_0 \log \frac{4}{\delta}} + \frac{\epsilon}{1 - \alpha \sigma_{H_0}}$. That is

$$\mathbb{P}\left(\left\|\nabla \widehat{J}(\widehat{\theta}_{k+1})\right\|_{2} \le 2\alpha C_{1}\sqrt{\widehat{\kappa}_{0}\log\frac{4}{\delta}} + \frac{\epsilon}{1-\sqrt{\alpha}\sigma_{H_{0}}}\right) \ge 1-\delta.$$
(92)

This concludes the proof.

For simplify, we introduce two following notations C_1, C_2 to short expressions,

$$C_2 = 2C_1 \sqrt{\widehat{\kappa}_0 \log \frac{4}{\delta}} + \frac{\epsilon}{1 - \sqrt{\alpha} \sigma_{H_0}}, C_3 = 2C_1 \sqrt{\widehat{\kappa}_0 \log \frac{4}{\delta}} + \frac{1}{1 - \sqrt{\alpha} \sigma_{H_0}}.$$
(93)

E.6 Proof of Lemma 11

Firstly, let's bound on the term $\|\nabla A^{\pi_{\theta}}(s,a)\|_2 = \|\nabla Q^{\pi_{\theta}}(s,a) - \nabla V^{\pi_{\theta}}(s)\|_2$. In fact,

$$\|\nabla V^{\pi_{\theta}}(s)\|_{2} = \left\|\nabla \mathbb{E}_{\pi_{\theta}}\left[\sum_{t=0}^{\infty} \gamma^{t} r_{t+1}\right]\right\|_{2} = \left\|\sum_{a \in \mathcal{A}} \underbrace{\nabla \pi_{\theta}(a|s)}_{\substack{(7)\\ \leq U}} \sum_{t=0}^{\infty} \gamma^{t} r_{t+1}\right\|_{2} \leq \sum_{a \in \mathcal{A}} \frac{R_{\max}}{1-\gamma} = \frac{UR_{\max}|\mathcal{A}|}{1-\gamma},$$
(94)

$$\|\nabla Q^{\pi_{\theta}}(s,a)\|_{2} = \left\|\nabla \left(R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a) V^{\pi_{\theta}}(s)\right)\right\|_{2} = \gamma \left\|\sum_{s' \in \mathcal{S}} P(s'|s,a) \nabla V^{\pi_{\theta}}(s)\right\|_{2}$$
$$\leq \gamma \sum_{s' \in \mathcal{S}} \|\nabla V^{\pi_{\theta}}(s)\|_{2} \stackrel{(94)}{\leq} \frac{\gamma U R_{\max}|\mathcal{A}||\mathcal{S}|}{1-\gamma}.$$
(95)

Thus, by the results of (94) and (95), we have

$$\left\|\nabla A^{\pi_{\theta}}(s,a)\right\|_{2} \leq \left\|\nabla Q^{\pi_{\theta}}(s,a)\right\|_{2} + \left\|\nabla V^{\pi_{\theta}}(s)\right\|_{2} \leq (1+\gamma|\mathcal{S}|)\frac{UR_{\max}|\mathcal{A}|}{1-\gamma}.$$
(96)

Now, we consider the boundedness of $\left\|\nabla J(\theta_{\star}) - \nabla J(\theta_k)\right\|_2$, in fact,

$$\begin{aligned} \left\| \nabla \left(J(\theta_{k}) - J(\theta_{\star}) \right) \right\|_{2} \stackrel{(26)}{=} \left\| \nabla \int_{s \in \mathcal{S}} \int_{a \in \mathcal{A}} d_{\rho_{0}}^{\pi_{\theta_{\star}}}(s) \pi_{\theta_{\star}}(a|s) A^{\pi_{\theta}}(s, a) \mathrm{d}s \mathrm{d}a \right\|_{2} \\ &= \left\| \nabla \int_{s \in \mathcal{S}} \int_{a \in \mathcal{A}} \frac{d_{\rho_{0}}^{\pi_{\theta_{\star}}}(s)}{d_{\rho_{0}}^{\pi_{\theta}}(s)} d_{\rho_{0}}^{\pi_{\theta}}(s) \pi_{\theta_{\star}}(a|s) A^{\pi_{\theta}}(s, a) \mathrm{d}s \mathrm{d}a \right\|_{2} \\ &= \left\| \int_{s \in \mathcal{S}} \int_{a \in \mathcal{A}} \frac{d_{\rho_{0}}^{\pi_{\theta_{\star}}}(s)}{d_{\rho_{0}}^{\pi_{\theta}}(s)} d_{\rho_{0}}^{\pi_{\theta}}(s) \underbrace{\pi_{\theta_{\star}}(a|s)}_{\leq 1} \nabla A^{\pi_{\theta}}(s, a) \mathrm{d}s \mathrm{d}a \right\|_{2} \\ &\leq \int_{s \in \mathcal{S}} \int_{a \in \mathcal{A}} \left(\frac{d_{\rho_{0}}^{\pi_{\theta_{\star}}}(s)}{d_{\rho_{0}}^{\pi_{\theta}}(s)} \| d_{\rho_{0}}^{\pi_{\theta}}(s) \|_{2} \| \nabla A^{\pi_{\theta}}(s, a) \|_{2} \right) \mathrm{d}s \mathrm{d}a \\ &\leq \int_{s \in \mathcal{S}} \int_{a \in \mathcal{A}} \underbrace{\max_{s \in \mathcal{S}} \frac{d_{\rho_{0}}^{\pi_{\theta_{\star}}}(s)}{d_{\rho_{0}}^{\pi_{\theta}}(s)}}_{= \left\| \frac{d_{\rho_{0}}^{\pi_{\theta_{\star}}}}{d_{\rho_{0}}^{\pi_{\theta_{\star}}}} \right\|_{\infty}^{\pi_{\theta_{\star}}} (1 + \gamma|\mathcal{S}|) \frac{UR_{\max}|\mathcal{A}|}{(1 - \gamma)^{2}} \mathrm{d}s \mathrm{d}a \tag{98}$$

$$= \left\| \frac{d_{\rho_0}^{\pi_{\theta_{\star}}}}{\rho_0} \right\|_{\infty} \frac{(1+\gamma|\mathcal{S}|)|\mathcal{A}|^2|\mathcal{S}|UR_{\max}}{(1-\gamma)^2},$$
(99)

where Eq.(97) holds since

$$\|d_{\rho_0}^{\pi_{\theta}}(s)\|_2 = \|\mathbb{E}_{s_0 \sim \rho_0(\cdot)}[d_{s_0}^{\pi_{\theta}}(s)]\|_2 = \|\mathbb{E}_{s_0 \sim \rho_0(\cdot)}[\sum_{t=0}^{\infty} \gamma^t \underbrace{P^{\pi_{\theta}}(s_t = s|s_0)}_{\leq 1}]\|_2 \leq \frac{1}{1-\gamma}.$$

Eq.(98) holds since $d_{\rho_0}^{\pi_{\theta}}(s) \ge \rho_0(s)$, thus, we have

$$\left\|\frac{d_{\rho_0}^{\pi_{\theta_\star}}}{d_{\rho_0}^{\pi_{\theta}}}\right\|_{\infty} = \max_{s \in \mathcal{S}} \frac{d_{\rho_0}^{\pi_{\theta_\star}}(s)}{d_{\rho_0}^{\pi_{\theta}}(s)} \le \max_{s \in \mathcal{S}} \frac{d_{\rho_0}^{\pi_{\theta_\star}}(s)}{\rho_0(s)} = \left\|\frac{d_{\rho_0}^{\pi_{\theta_\star}}}{\rho_0}\right\|_{\infty}$$

E.7 Proof of Lemma 12

Organization It is very technical to achieve the results of Lemma 12, our outline our proof as follows: in section E.7.1, we bound all the four terms of (68); in section E.7.2, we bound the term Z_j with high probability, which is a preliminary result for applying Azuma's Inequality to get a further result; in section E.7.3, we bound the $\Delta_k = \nabla J(\theta_k) - \nabla \widehat{J}(\widehat{\theta}_k)$ with high probability; finally, in section E.7.4, we bound $\theta_{k+1} - \widehat{\theta}_{k+1}$ with high probability.

Proof. (of Lemma 12) Before we present the details of the proof, we need some preliminary results. Preliminary results. Under Assumption 1-2, we have

$$\|\Delta H_k\|_{op} \le \chi (\|\theta_{k+1} - \widehat{\theta}_{k+1}\|_2 + \|\widehat{\theta}_{k+1} - \widehat{\theta}_0\|_2), \quad \|e_k\|_2 \le \frac{\chi}{2} \|\theta_k - \theta_{k+1}\|_2.$$
(100)

We define two events \mathcal{B}_k and \mathcal{C}_k as follows,

$$\mathcal{B}_{k} = \bigcap_{j=0}^{k} \left\{ j \le k : \left\| \widehat{\theta}_{j} - \widehat{\theta}_{0} \right\|_{2} \le \alpha C_{2} \text{ and } \left\| \nabla \widehat{J}(\widehat{\theta}_{j}) \right\|_{2} \le \alpha C_{3} \right\},$$
(101)

$$C_{k} = \bigcap_{j=0}^{k} \left\{ j \le k : \max\left\{ \left\| \Delta_{j} \right\|_{2}, \left\| \Delta_{j} \right\|_{2}^{2} \right\} \le C_{4} \sqrt{\alpha^{3} \log \frac{1}{\alpha}} \right\},$$
(102)

where C_2 and C_3 are defined in (93), C_4 is a positive constant, and we will provide the rule of choosing C_4 in (123). Since $\Delta_0 = 0$, the event $C_0 \neq \emptyset$. The chosen rule of C_4 ensures for each $k \in [0, \hat{\kappa}_0]$, the event $C_k \neq \emptyset$.

Furthermore, for each $k \in [0, \hat{\kappa}_0]$, the following equation occurs with probability at least $1 - \delta$,

$$\|\Delta H_k\|_{op} \stackrel{(100)}{\leq} \chi\left(\underbrace{\|\theta_{k+1} - \widehat{\theta}_{k+1}\|_2}_{\substack{(69)\\\leq \alpha}\|\sum_{j=0}^k \Delta_j\|_2} + \underbrace{\|\widehat{\theta}_{k+1} - \widehat{\theta}_0\|_2}_{\substack{(101)\\\leq \alpha C_2}}\right)$$
$$\leq \chi\alpha\left(\sum_{j=0}^k \|\Delta_j\|_2 + C_2\right) \stackrel{(102)}{\leq} \chi\alpha\left(C_4\widehat{\kappa}_0\sqrt{\alpha^3\log\frac{1}{\alpha}} + C_2\right). \tag{103}$$

E.7.1 Bounded of all the four terms of Eq.(68)

For each $k \in [0, \hat{\kappa}_0]$, the following equation occurs with probability at least $1 - \delta$:

1 for the first term of Eq.(68),

$$\left\| (I + \alpha H_0) \Delta_k \right\|_2 \le \left\| I + \alpha H_0 \right\|_{op} \left\| \Delta_k \right\|_2 \stackrel{(102)}{\le} (1 + \alpha \Lambda) \sqrt{\alpha^3 \log \frac{1}{\alpha}} C_4 =: \sqrt{\alpha^3 \log \frac{1}{\alpha}} B_1, \quad (104)$$

where $B_1 = (1 + \alpha \Lambda)C_4$.

2 for the second term of Eq.(68),

$$\begin{aligned} \left\| \alpha \Delta H_k [\Delta_k + \nabla \widehat{J}(\widehat{\theta}_k)] \right\|_2 &= \alpha \left\| \Delta H_k \nabla J(\theta_k) \right\|_2 \le \alpha \left\| \Delta H_k \right\|_{op} \left\| \nabla J(\theta_k) \right\|_2 \\ &\stackrel{(99),(103)}{\le} \alpha^2 \chi \Big(C_4 \widehat{\kappa}_0 \sqrt{\alpha^3 \log \frac{1}{\alpha}} + C_2 \Big) \left\| \frac{d_{\rho_0}^{\pi_{\theta_*}}}{\rho_0} \right\|_\infty \frac{(1 + \gamma |\mathcal{S}|) |\mathcal{A}|^2 |\mathcal{S}| U R_{\max}}{(1 - \gamma)^2} \\ &=: \alpha^2 B_2, \end{aligned}$$
(105)

where $B_2 = \chi \left(C_4 \hat{\kappa}_0 \sqrt{\alpha^3 \log \frac{1}{\alpha}} + C_2 \right) \left\| \frac{d_{\rho_0}^{\pi_{\theta_{\star}}}}{\rho_0} \right\|_{\infty} \frac{(1+\gamma|\mathcal{S}|)|\mathcal{A}|^2|\mathcal{S}|UR_{\max}}{(1-\gamma)^2}.$ **3** for the third term of Eq.(68),

$$\begin{aligned} \|\alpha \Delta H_k(\widehat{\xi}_k + \xi_0)\|_2 \\ &\leq \alpha^2 \chi \Big(\sqrt{\alpha^3 \log \frac{1}{\alpha}} \widehat{\kappa}_0 C_4 + C_2\Big) \Big(\frac{\sigma_{H_0} G R_{\max}}{1 - \gamma} \Big(\frac{1}{\sqrt{\chi \epsilon} (1 - \sqrt{\alpha} \sigma_{H_0})} + \frac{\alpha}{1 - \alpha \sigma_{H_0}}\Big) + \sigma\Big) \\ &=: \alpha^2 B_3, \end{aligned}$$
(106)

where $B_3 = \chi \left(\sqrt{\alpha^3 \log \frac{1}{\alpha}} \widehat{\kappa}_0 C_4 + C_2 \right) \left(\frac{\sigma_{H_0} G R_{\max}}{1 - \gamma} \left(\frac{1}{\sqrt{\chi \epsilon} (1 - \sqrt{\alpha} \sigma_{H_0})} + \frac{\alpha}{1 - \alpha \sigma_{H_0}} \right) + \sigma \right).$

 \bullet for the forth term of Eq.(68),

$$\|e_k\|_2 = \|\theta_{k+1} - \theta_k\|_2 \le \alpha \frac{\chi}{2} \|g(\theta_k)\|_2 \stackrel{(44)}{\le} \frac{1}{2} \alpha \chi \Big(\frac{GR_{\max}}{1 - \gamma}\Big) =: \alpha B_4, \tag{107}$$

where $B_4 = \chi \left(\frac{GR_{\max}}{2(1-\gamma)} \right)$.

From the results of (104)-(107), we have

$$\mathbb{E}\Big[\|\Delta_{k+1}\|_{2}^{2}\Big|\mathcal{F}_{k}\Big] \stackrel{(68)}{\leq} \|(I+\alpha H_{0})\Delta_{k}\|_{2}^{2} + 2\alpha\|(I+\alpha H_{0})\Delta_{k}\|_{2}(\alpha B_{2}+\alpha B_{3}+B_{4}) \\ + \alpha^{4}B_{2}^{2} + \alpha^{4}B_{3}^{2} + \alpha^{2}B_{4}^{2} + 2(\alpha^{4}B_{2}B_{3}+\alpha^{3}B_{2}B_{4}+\alpha^{3}B_{3}B_{4}).$$
(108)

For simplify, we introduce two following notations to short expressions,

$$D_1 = \alpha B_2 + \alpha B_3 + B_4, \quad D_2 = \alpha^2 B_2^2 + \alpha^2 B_3^2 + B_4^2 + 2(\alpha^2 B_2 B_3 + \alpha B_2 B_4 + \alpha B_3 B_4), \quad (109)$$

then we can rewrite (108) as follows,

$$\mathbb{E}\Big[\|\Delta_{k+1}\|_{2}^{2} |\mathcal{F}_{k}\Big] \leq \|(I+\alpha H_{0})\Delta_{k}\|_{2}^{2} + 2\alpha D_{1} \|(I+\alpha H_{0})\Delta_{k}\|_{2} + \alpha^{2} D_{2} \\ \stackrel{(104)}{\leq} (1+\alpha\Lambda)^{2} \|\Delta_{k}\|_{2}^{2} + 2\alpha^{\frac{5}{2}} \sqrt{\log\frac{1}{\alpha}} D_{1} B_{1} + \alpha^{2} D_{2}.$$
(110)

Rearranging Eq.(110), we have

$$\mathbb{E}\left[\left\|\Delta_{k+1}\right\|_{2}^{2} + \frac{2\alpha^{\frac{3}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha D_{2}}{\alpha\Lambda^{2} + 2\Lambda}\right|\mathcal{F}_{k}\right]$$
$$\leq (1 + \alpha\Lambda)^{2}\left(\left\|\Delta_{k}\right\|_{2}^{2} + \frac{2\alpha^{\frac{3}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha D_{2}}{\alpha\Lambda^{2} + 2\Lambda}\right).$$
(111)

E.7.2 Boundedness of $Z_j = (1 + \alpha \Lambda)^{-2j} \Big(\|\Delta_j\|_2^2 + \frac{2\alpha^{\frac{3}{2}}\sqrt{\log\frac{1}{\alpha}}D_1B_1 + \alpha D_2}{\alpha\Lambda^2 + 2\Lambda} \Big).$

We will use Azuma's inequality, i.e., Lemma 1 to achieve the boundedness of Z_j , which require us to show: (i) $\{Z_j\}$ is a super-martingale, (ii) the boundedness $|Z_{j+1} - Z_j|$ almost surely.

Let $\mathcal{F}_k = \sigma\{\xi_0, \hat{\xi}_1, \dots, \hat{\xi}_k\}$ be a filtration generated by all the information from time 0 to k. Then

 $\{Z_j\}$ is a super-martingale with respect to the filtration $\{\mathcal{F}_j\}$. In fact, for each $j \in [0, \hat{\kappa}_0]$, we have

$$\mathbb{E}\left[Z_{j}\middle|\mathcal{F}_{j-1}\right] = (1 + \alpha\Lambda)^{-2j} \left(\mathbb{E}\left[\|\Delta_{j}\|_{2}^{2}\right] + \frac{2\alpha^{\frac{3}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha D_{2}}{\alpha\Lambda^{2} + 2\Lambda}\right) \\
\stackrel{(110)}{\leq} (1 + \alpha\Lambda)^{-2j} \left((1 + \alpha\Lambda)^{2}\|\Delta_{j-1}\|_{2}^{2} + 2\alpha^{\frac{5}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha^{2}D_{2} + \frac{2\alpha^{\frac{3}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha D_{2}}{\alpha\Lambda^{2} + 2\Lambda}\right) \\
\leq (1 + \alpha\Lambda)^{-2(j-1)} \left(\|\Delta_{j-1}\|_{2}^{2} + \underbrace{\frac{2\alpha^{\frac{5}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha^{2}D_{2}}{(1 + \alpha\Lambda)^{2}} + \frac{2\alpha^{\frac{3}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha D_{2}}{(\alpha\Lambda^{2} + 2\Lambda)(1 + \alpha\Lambda)^{2}}\right) \\
= \frac{2\alpha^{\frac{3}{2}}\sqrt{\log\frac{1}{\alpha}}D_{1}B_{1} + \alpha D_{2}}{\alpha\Lambda^{2} + 2\Lambda} \tag{112}$$

which implies Z_j is a super-martingale.

We introduce β to short the expression Z_j ,

$$\beta = \frac{2\sqrt{\alpha \log \frac{1}{\alpha}}D_1B_1 + D_2}{\Lambda^2 + 2\Lambda}$$

i.e., $Z_j = (1 + \alpha \Lambda)^{-2j} (\|\Delta_j\|_2^2 + \alpha \beta).$

Furthermore, let us bound the term $|Z_j - Z_{j-1}|$. In fact, since Z_j is a super-martingale, then

$$|Z_{j} - Z_{j-1}|$$

$$\leq |Z_{j} - \mathbb{E}[Z_{j}|\mathcal{F}_{j-1}]| = (1 + \alpha \Lambda)^{-2j} |||\Delta_{j}^{2}||_{2} - \mathbb{E}[||\Delta_{j}^{2}||_{2}|\mathcal{F}_{j-1}]|.$$
(113)

The result of (113) shows that to bound the term $|Z_j - Z_{j-1}|$, we need to bound the term $|||\Delta_j^2||_2 - \mathbb{E}[\Delta_j^2||_2|\mathcal{F}_{j-1}]|$.

Recall the filtration $\mathcal{F}_j = \sigma\{\xi_0, \hat{\xi}_1, \dots, \hat{\xi}_j\}$, and Δ_j defined in (68), where the first two terms are deterministic, and only the third and fourth term are random. Then the following holds almost surely,

$$\begin{aligned} \left| \|\Delta_{j}\|_{2}^{2} - \mathbb{E} \Big[\|\Delta_{j}\|_{2}^{2} |\mathcal{F}_{j-1} \Big] \right| &\leq 4\alpha \| (I + \alpha H_{0}) \Delta_{j} \|_{2} (\alpha B_{3} + B_{4}) \\ &+ 2\alpha^{4} B_{3}^{2} + 2\alpha^{2} B_{4}^{2} + 4 (\alpha^{4} B_{2} B_{3} + \alpha^{3} B_{2} B_{3} + \alpha^{3} B_{3} B_{4}) \\ &\leq 4\alpha \sqrt{\alpha^{3} \log \frac{1}{\alpha}} B_{1} (\alpha B_{3} + B_{4}) \\ &+ 2\alpha^{4} B_{3}^{2} + 2\alpha^{2} B_{4}^{2} + 4 (\alpha^{4} B_{2} B_{3} + \alpha^{3} B_{2} B_{3} + \alpha^{3} B_{3} B_{4}) \\ &=: c_{j}. \end{aligned}$$

$$(114)$$

To short the expression, we introduce two notations as follows,

$$E_1 = 4B_1(\alpha B_3 + B_4),$$

$$E_2 = 2\alpha^2 B_3^2 + 2B_4^2 + 4(\alpha^2 B_2 B_3 + \alpha B_2 B_3 + \alpha B_3 B_4).$$
(115)

From the results of (113) and (114), we have the boundedness of $|Z_j - Z_{j-1}|$ as follows,

$$|Z_j - Z_{j-1}| \le \alpha^{\frac{5}{2}} \sqrt{\log \frac{1}{\alpha}} E_1 + \alpha^2 E_2 = c_j.$$

Finally, from Azuma's inequality, i.e., Lemma 1, the following holds, for any $\delta > 0$, we have

$$\mathbb{P}(Z_{\widehat{\kappa}_0} - Z_0 \ge \delta) \le \exp\left(-\frac{2\delta^2}{\sum_{j=0}^{\widehat{\kappa}_0} c_j^2}\right),$$

which implies the following holds with probability less than α^2

$$Z_{\widehat{\kappa}_0} - Z_0 \ge \sqrt{\log \frac{1}{\alpha} \sum_{j=0}^{\widehat{\kappa}_0} c_j^2}.$$
(116)

Recall $\hat{\kappa}_0 = \left\lfloor \frac{\log\left(\frac{1}{1-\sqrt{\alpha\sigma_{H_0}}}\right)}{\log(1+\alpha\sqrt{\chi\epsilon})} \right\rfloor$, and Taylor's expression of $\log(1+x)$, $r^2 = r^3 = r^4$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n} + \dots, \quad x \in (-1,1),$$

then for a enough small α , $\hat{\kappa}_0$ is upper-bounded as follows,

$$\widehat{\kappa}_{0} = \left\lfloor \frac{\log\left(\frac{1}{1 - \sqrt{\alpha}\sigma_{H_{0}}}\right)}{\log(1 + \alpha\sqrt{\chi\epsilon})} \right\rfloor = \mathcal{O}\left(\frac{\sqrt{\alpha}\sigma_{H_{0}}}{\alpha\Lambda}\right) + 1 \le \mathcal{O}\left(\frac{\sigma_{H_{0}}}{\sqrt{\chi\epsilon\alpha}}\right).$$
(117)

From the result of (116), for each $k \in [1, \hat{\kappa}_0]$, we have

$$\alpha^2 \ge \mathbb{P}\left(Z_k - Z_0 \ge \sqrt{\log\frac{1}{\alpha}\sum_{j=0}^{\hat{\kappa}_0} c_j^2}\right) = \mathbb{P}\left(Z_k \ge Z_0 + \sqrt{\log\frac{1}{\alpha}\sum_{j=0}^{\hat{\kappa}_0} c_j^2}\right)$$
(118)

$$= \mathbb{P}\left(Z_k \ge \alpha\beta + \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_2\sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right)\right),\tag{119}$$

where Eq.(119) holds from (118) due to the following three aspects:

(i) $Z_0 = \|\Delta_0\|_2^2 + \alpha\beta$ and $\Delta_0 = \nabla J(\theta_0) - \nabla \widehat{J}(\widehat{\theta}_0) \stackrel{(61)}{=} 0$, which implies $Z_0 = \alpha\beta$;

(ii) recall $c_j = \alpha^{\frac{5}{2}} \sqrt{\log \frac{1}{\alpha}} E_1 + \alpha^2 E_2$, which implies for an enough small α , we can estimate c_j as follows, $c_j^2 = \mathcal{O}\left(\alpha^4 E_2^2\right)$;

(iii)furthermore, since $\hat{\kappa}_0$ is upper bounded by $\mathcal{O}\left(\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon\alpha}}\right)$, then we have

$$\sqrt{\log \frac{1}{\alpha} \sum_{j=0}^{\widehat{\kappa}_0} c_j^2} = \mathcal{O}\left(\alpha^{\frac{7}{4}} \sqrt{\log \frac{1}{\alpha}} E_2 \sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right)$$

Rewrite (119), we present the boundedness of Z_k with high probability as follows,

$$\mathbb{P}\left(Z_k \le \alpha\beta + \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_2\sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right)\right) \ge 1 - \alpha^2.$$

E.7.3 Boundedness of $\Delta_k = \nabla J(\theta_k) - \nabla \widehat{J}(\widehat{\theta}_k)$.

Recall $Z_k = (1 + \alpha \Lambda)^{-2k} (\|\Delta_k\|_2^2 + \alpha \beta)$, the event

$$Z_k \ge \alpha\beta + \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_2\sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right)$$

is equivalent to

$$\|\Delta_k\|_2^2 \ge (1+\alpha\Lambda)^{2k} \cdot \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_2\sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right) + \left((1+\alpha\Lambda)^{2k}-1\right)\alpha\beta.$$
(120)

Since $\hat{\kappa}_0$: $(1 + \alpha \Lambda)^{\hat{\kappa}_0} \leq \frac{1}{1 - \sqrt{\alpha} \sigma_{H_0}}$, which implies the first term of (120) is upper-bounded as follows,

$$(1+\alpha\Lambda)^{2k} \cdot \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_2\sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right) \leq \left(\frac{1}{1-\sqrt{\alpha}\sigma_{H_0}}\right)^2 \cdot \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_2\sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right).$$
(121)

The second term of (120) is upper-bounded as follows: for a small α , for each $k \in [0, \hat{\kappa}_0]$,

$$\begin{pmatrix} (1+\alpha\Lambda)^{2k}-1 \end{pmatrix} \alpha\beta = \begin{pmatrix} \sum_{j=0}^{2k} \binom{2k}{j} (\alpha\Lambda)^j & -1 \end{pmatrix} \alpha\beta$$

Newton's binomial expression of $(1+\alpha\Lambda)^{2k}$

$$= \begin{pmatrix} 2k\alpha\Lambda + \frac{2k(2k-1)(\alpha\Lambda)^2}{2} + \dots + (\alpha\Lambda)^{2k} \end{pmatrix} \alpha\beta$$

$$= \mathcal{O}\Big(2k\alpha^2\Lambda\beta\Big) \stackrel{(117)}{\leq} \mathcal{O}\Big(\frac{2\alpha^{\frac{3}{2}}\Lambda\beta\sigma_{H_0}}{\sqrt{\chi\epsilon}}\Big).$$
(122)

Combining above (121), (122), we can bound the right equation of (120) as follows,

$$(1 + \alpha \Lambda)^{2k} \cdot \mathcal{O}\left(\alpha^{\frac{7}{4}} \sqrt{\log \frac{1}{\alpha}} E_2 \sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right) + \left((1 + \alpha \Lambda)^{2k} - 1\right) \alpha \beta$$

$$\leq \mathcal{O}\left(\alpha^{\frac{7}{4}} \sqrt{\log \frac{1}{\alpha}} E_2 \sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right) + \mathcal{O}\left(\frac{2\alpha^{\frac{3}{2}}\Lambda\beta\sigma_{H_0}}{\sqrt{\chi\epsilon}}\right) + \left(\frac{1}{1 - \sqrt{\alpha}\sigma_{H_0}}\right)^2$$

$$\leq \mathcal{O}\left(\max\left\{\frac{2\alpha^{\frac{3}{2}}\Lambda\beta\sigma_{H_0}}{\sqrt{\chi\epsilon}}, \alpha^{\frac{7}{4}} \sqrt{\log \frac{1}{\alpha}} E_2 \sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right\}\right)$$

$$\leq C_4 \sqrt{\alpha^3 \log \frac{1}{\alpha}}.$$
(123)

The last (123) present the condition of the parameter C_4 . Now, we define two events $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ as follows,

$$\widetilde{\mathcal{E}}_1 = \left\{ \|\Delta_k\|_2^2 \ge (1 + \alpha \Lambda)^{2k} \cdot \mathcal{O}\left(\alpha^{\frac{7}{4}} \sqrt{\log \frac{1}{\alpha}} E_2 \sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right) + \left((1 + \alpha \Lambda)^{2k} - 1\right) \alpha \beta \right\}$$
$$\widetilde{\mathcal{E}}_2 = \left\{ \|\Delta_k\|_2^2 \ge C_4 \sqrt{\alpha^3 \log \frac{1}{\alpha}} \right\}.$$

The results of (120) and (123) imply the event $\widetilde{\mathcal{E}}_1$ contains $\widetilde{\mathcal{E}}_2$, i.e., $\widetilde{\mathcal{E}}_2 \subset \widetilde{\mathcal{E}}_1$. Then

$$\mathbb{P}(\widetilde{\mathcal{E}}_{2}) = \mathbb{P}\left(\|\Delta_{k}\|_{2}^{2} \ge C_{4}\sqrt{\alpha^{3}\log\frac{1}{\alpha}}\right) \\
\le \mathbb{P}(\widetilde{\mathcal{E}}_{1}) \stackrel{(\mathbf{a})}{=} \mathbb{P}\left(Z_{k} \ge \alpha\beta + \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_{2}\sqrt{\frac{\sigma_{H_{0}}}{\sqrt{\chi\epsilon}}}\right)\right) \stackrel{(119)}{\le} \alpha^{2},$$
(124)

where (a) of (124) holds since the event $\widetilde{\mathcal{E}}_1$ is equivalent to $\left\{Z_k \geq \alpha\beta + \mathcal{O}\left(\alpha^{\frac{7}{4}}\sqrt{\log\frac{1}{\alpha}}E_2\sqrt{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}}\right)\right\}$, we have provided the detail of this conclusion in (120).

Furthermore, for each $k \in [0, \hat{\kappa}_0]$, we have

$$\overline{\mathcal{C}_{k}} \overset{(102)}{\subseteq} \overline{\bigcap_{j=0}^{k} \left\{ j \leq k : \|\Delta_{j}\|_{2}^{2} \leq C_{4} \sqrt{\alpha^{3} \log \frac{1}{\alpha}} \right\}} = \bigcup_{j=0}^{k} \overline{\left\{ j \leq k : \|\Delta_{j}\|_{2}^{2} \leq C_{4} \sqrt{\alpha^{3} \log \frac{1}{\alpha}} \right\}} \\
= \underbrace{\bigcup_{j=0}^{k-1} \overline{\left\{ j \leq k : \|\Delta_{j}\|_{2}^{2} \leq C_{4} \sqrt{\alpha^{3} \log \frac{1}{\alpha}} \right\}}}_{=\overline{\mathcal{C}_{k-1}}} \bigcup \left\{ \|\Delta_{k}\|_{2}^{2} \geq C_{4} \sqrt{\alpha^{3} \log \frac{1}{\alpha}} \right\} \\
= \overline{\mathcal{C}_{k-1}} \cup \widetilde{\mathcal{E}}_{2},$$

which implies for each $k \in [0, \hat{\kappa}_0]$,

$$\mathbb{P}(\overline{\mathcal{C}_k}) \le \mathbb{P}(\overline{\mathcal{C}_{k-1}}) + \mathbb{P}(\widetilde{\mathcal{E}}_2) \le \mathbb{P}(\overline{\mathcal{C}_{k-1}}) + \alpha^2.$$

Summing the above equation from k = 1 to $\hat{\kappa}_0$, we have

$$\mathbb{P}(\overline{\mathcal{C}_{\widehat{\kappa}_{0}}}) = \underbrace{\mathbb{P}(\overline{\mathcal{C}_{0}})}_{=0} + \sum_{j=1}^{\widehat{\kappa}_{0}} \left(\mathbb{P}(\overline{\mathcal{C}_{j}}) - \mathbb{P}(\overline{\mathcal{C}_{j-1}}) \right)$$
$$\leq \widehat{\kappa}_{0} \alpha^{2} \leq \left\lfloor \frac{\log\left(\frac{1}{1-\sqrt{\alpha}\sigma_{H_{0}}}\right)}{\log(1+\alpha\sqrt{\chi\epsilon})} \right\rfloor \alpha^{2} = \frac{\sigma_{H_{0}}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}} + o(\alpha^{\frac{3}{2}}), \tag{125}$$

where $\mathbb{P}(\overline{\mathcal{C}_0}) = 0$ since the event $\overline{\mathcal{C}_0} = \left\{ \Delta_0 = 0 > C_4 \sqrt{\alpha^3 \log \frac{1}{\alpha}} \right\}$ can not occur. The result of (125) shows that

$$\mathbb{P}\left(\max\left\{\|\Delta_k\|_2, \|\Delta_k\|_2^2\right\} \ge C_4 \sqrt{\alpha^3 \log \frac{1}{\alpha}}\right) \le \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}} + o(\alpha^{\frac{3}{2}}).$$

E.7.4 Boundedness of $\theta_{k+1} - \hat{\theta}_{k+1}$ (69).

From the previous result, the following happens at less than $1 - \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}}$

$$\begin{aligned} \left\|\theta_{k+1} - \widehat{\theta}_{k+1}\right\|_{2} & \stackrel{(69)}{\leq} \alpha \sum_{j=0}^{k} \|\Delta_{j}\|_{2} \leq \alpha \widehat{\kappa}_{0} C_{4} \sqrt{\alpha^{3} \log \frac{1}{\alpha}} = \alpha \sqrt{\alpha^{3} \log \frac{1}{\alpha}} \left\lfloor \frac{\log\left(\frac{1}{1 - \sqrt{\alpha}\sigma_{H_{0}}}\right)}{\log(1 + \alpha\sqrt{\chi\epsilon})} \right\rfloor C_{4} \\ &= \alpha^{2} \sqrt{\log \frac{1}{\alpha}} \frac{\sigma_{H_{0}}}{\sqrt{\chi\epsilon}} C_{4} + o\left(\alpha^{2} \sqrt{\log \frac{1}{\alpha}}\right). \end{aligned}$$
(126)

This concludes the proof.

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E.8 Proof of Proposition 2

Recall Assumption 2: $H(\theta) =: \nabla^2 J(\theta)$ is χ -Hessian-Lipschitz, then for any θ , θ_0 , we have

$$J(\theta) \ge J(\theta_0) + \nabla J(\theta_0)^{\top} (\theta - \theta_0) + \frac{1}{2} (\theta - \theta_0)^{\top} H_0(\theta - \theta_0) - \frac{\chi}{6} \|\theta_0 - \theta\|_2^3.$$
(127)

Let $\hat{\phi} = \hat{\theta}_{k+1} - \theta_0 = \hat{\theta}_{k+1} - \hat{\theta}_0$ and $\phi = \theta_{k+1} - \hat{\theta}_{k+1}$, after some careful calculations, we can rewrite above equation (127) as follows,

$$J(\theta_{k+1}) - J(\theta_0) \ge J_1 + J_2,$$

where $J_1 = \nabla J(\theta_0)^\top \widehat{\phi} + \frac{1}{2} \widehat{\phi}^\top H_0 \widehat{\phi}, \quad J_2 = \nabla J(\theta_0)^\top \phi + \widehat{\phi}^\top H_0 \phi + \frac{1}{2} \phi^\top H_0 \phi - \frac{\chi}{6} \|\widehat{\phi} + \phi\|_2^3$. Firstly, we show the lower-boundedness of the expectation of J_1 .

Since $J_1 = \nabla J(\theta_0)^\top \widehat{\phi} + \frac{1}{2} \widehat{\phi}^\top H_0 \widehat{\phi}$, we bound $\nabla J(\theta_0)^\top \widehat{\phi}$ at first, then bound $\frac{1}{2} \widehat{\phi}^\top H_0 \widehat{\phi}$.

Recall the result of (65), since $\hat{\theta}_0 = \theta_0$, then we have

$$\nabla J(\theta_0)^{\top} \widehat{\phi} = \nabla J(\theta_0)^{\top} (\widehat{\theta}_{k+1} - \theta_0)$$
$$= \alpha \sum_{j=0}^k \nabla J(\widehat{\theta}_0)^{\top} (I + \alpha H_0)^j \nabla J(\widehat{\theta}_0) + \alpha \sum_{j=0}^k \nabla J(\widehat{\theta}_0)^{\top} (I + \alpha H_0)^{k-j} (\widehat{\xi}_j + \xi_0).$$
(128)

Furthermore, there exists an orthogonal matrix $Q \in \mathbb{R}^{p \times p}$ (i.e., $Q^{\top}Q = I$), s.t., $Q^{\top}H_0Q = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p, \end{pmatrix} \Rightarrow Q^{\top}(I + \alpha H_0)^j Q = \begin{pmatrix} (1 + \alpha \lambda_1)^j & 0 & \cdots & 0 \\ 0 & (1 + \alpha \lambda_2)^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & (1 + \alpha \lambda_p)^j \end{pmatrix}$. Let $Q = \begin{pmatrix} (1 + \alpha \lambda_1)^j & 0 & \cdots & 0 \\ 0 & (1 + \alpha \lambda_2)^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & (1 + \alpha \lambda_p)^j \end{pmatrix}$.

 (u_1, u_2, \dots, u_p) , then we have $(I + \alpha H_0)^j = \sum_{i=1}^p (1 + \alpha \lambda_i)^j u_i u_i^\top$. In fact, with some simple linear algebra, we know u_i is the unit eigenvector with respect to the eigenvalue λ_i correspondingly. Since then, we can rewrite the first term of (128) as follows,

$$\nabla J(\widehat{\theta}_0)^\top (I + \alpha H_0)^j \nabla J(\widehat{\theta}_0) = \sum_{i=1}^p (1 + \alpha \lambda_i)^j \nabla J(\widehat{\theta}_0)^\top u_i u_i^\top \nabla J(\widehat{\theta}_0) = \sum_{i=1}^p (1 + \alpha \lambda_i)^j e_i^2, \quad (129)$$

where $e_i =: u_i^\top \nabla J(\widehat{\theta}_0) = u_i^\top \nabla J(\theta_0) = u_i^\top \mathbb{E}[g(\theta_0)] = \mathbb{E}[u_i^\top g(\theta_0)]$, since we set the initial $\widehat{\theta}_0 = \theta_0$. Recall $H_0 u_p = \lambda_p u_p$, i.e., $u_p = \lambda_p^{-1} H_0 u_p$, and $||u_p||_2 = 1$, which implies $\lambda_p \leq ||H_0||_{op}$. Recall

$$g(\theta_{k}) =: g(\theta_{k}|\tau_{k}) = \sum_{t=0}^{h} \nabla \log \pi_{\theta}(a_{t}|s_{t})R(\tau_{k})|_{\theta=\theta_{k}}, \text{ then we have}$$

$$\mathbb{E}[e_{p}^{2}] = \mathbb{E}[(u_{p}^{\top}g(\theta_{0}))^{2}] = \mathbb{E}\Big[\left(u_{p}^{\top}\sum_{t=0}^{h} \nabla \log \pi_{\theta}(a_{t}|s_{t})R(\tau_{k})|_{\theta=\theta_{0}}\right)^{2}\Big]$$

$$\geq \frac{R_{\min}^{2}}{(1-\gamma)^{2}} \mathbb{E}\Big[\left(u_{p}^{\top}\sum_{t=0}^{h} \nabla_{\theta} \log \pi_{\theta}(a_{t}|s_{t})\right)^{2}\Big]$$

$$\geq \frac{R_{\min}^{2}}{(1-\gamma)^{2}} \mathbb{E}\Big(\Big[\sum_{t=0}^{h} (u_{p}^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{t}|s_{t}))^{2}\Big]$$

$$+ 2\mathbb{E}\Big[\sum_{0\leq i< j\leq h} u_{p}^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{i}|s_{i})u_{p}^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{j}|s_{j})\Big]\Big) \qquad (130)$$

$$= \frac{R_{\min}^{2}}{(1-\gamma)^{2}} \mathbb{E}\Big(\Big[\sum_{t=0}^{h} u_{p}^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{t}|s_{t})\nabla_{\theta}^{\top} \log \pi_{\theta}(a_{t}|s_{t})u_{p}\Big]$$

$$+ 2\mathbb{E}\Big[\sum_{0\leq i< j\leq h} u_{p}^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{i}|s_{i})u_{p}^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{j}|s_{j})\Big]\Big)$$

$$\geq \frac{R_{\min}^{2}h\omega}{(1-\gamma)^{2}} + 2\frac{R_{\min}^{2}}{(1-\gamma)^{2}||H_{0}||_{2p}^{2}} \mathbb{E}\Big[\sum_{0\leq i< j\leq h} (\lambda_{p}^{-1}H_{0}u_{p})^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{i}|s_{i})(\lambda_{p}^{-1}H_{0}u_{p})^{\top}\nabla_{\theta} \log \pi_{\theta}(a_{j}|s_{j})\Big]$$

$$\geq \frac{R_{\min}^{2}h\omega}{(1-\gamma)^{2}} + \frac{2R_{\min}^{2}\lambda_{p}^{2}}{(1-\gamma)^{2}||H_{0}||_{2p}^{2}} \mathbb{E}\Big[\sum_{0\leq i< j\leq h} \nabla_{\theta}^{\top} \log \pi_{\theta}(a_{i}|s_{i})\nabla_{\theta} \log \pi_{\theta}(a_{j}|s_{j})\Big], \qquad (131)$$

where Eq.(130) holds since a simple fact: $(\sum_{t=0}^{h} x_t)^2 = \sum_{t=0}^{h} x_t^2 + 2 \sum_{0 \le i < j \le h} x_i x_j$. **Discussion 1.** Now, we discuss the lower boundedness of (131), we introduce a notation c_0 as flows,

$$c_0 =: \mathbb{E}\Big[\sum_{0 \le i < j \le h} \nabla_{\theta}^{\top} \log \pi_{\theta}(a_i | s_i) \nabla_{\theta} \log \pi_{\theta}(a_j | s_j), \Big]$$

if $c_0 \ge 0$, the result of (131) shows that

$$\mathbb{E}[e_p^2] = \mathbb{E}\left[(u_p^\top g(\theta_0))^2\right] = \mathbb{E}[\langle g(\tau|\theta), u_p \rangle^2] \ge \frac{R_{\min}^2 h\omega}{(1-\gamma)^2};$$

if $c_0 < 0$, since $\lambda_p \ge \sqrt{\chi\epsilon}$, if we require (131) keep positive, then we have a small ϵ s.t.,

$$\epsilon \le \frac{h\omega \|H_0\|_{op}^2}{\chi |c_0|}.$$

Then, for a small enough ϵ , we have the lower boundedness of the term $\mathbb{E}[\langle g(\tau|\theta), u_p \rangle^2]$ as follows,

$$\mathbb{E}[\langle g(\tau|\theta), u_p \rangle^2] \ge \min\left\{\frac{R_{\min}^2 h\omega}{(1-\gamma)^2}, \frac{R_{\min}^2 h\omega}{(1-\gamma)^2} + \frac{2R_{\min}^2 \lambda_p^2}{(1-\gamma)^2 \|H_0\|_{op}^2} c_0\right\} =: \iota^2.$$
(132)

Now, we consider $\nabla J(\theta_0)^{\top} \widehat{\phi}$, for an enough small $0 < \alpha < \max\{\frac{1}{|\lambda_1|}, \frac{1}{\lambda_p}\}$, we have

$$\mathbb{E}[\nabla J(\theta_0)^\top \widehat{\phi}] \stackrel{(129)}{=} \alpha \sum_{j=0}^T \sum_{i=1}^p (1 + \alpha \lambda_i)^j \mathbb{E}[e_i^2] + \alpha \sum_{j=0}^T \nabla J(\widehat{\theta}_0)^\top (I + \alpha H_0)^{k-j} \mathbb{E}[(\widehat{\xi}_j + \xi_0)]$$
$$\geq \alpha \sum_{j=0}^T (1 + \alpha \lambda_p)^j \mathbb{E}[e_p^2] \geq \alpha (1 + \alpha \sqrt{\chi \epsilon}) \iota^2.$$
(133)

For the term $\frac{1}{2}\widehat{\phi}^{\top}H_0\widehat{\phi}$, we have,

$$\frac{1}{2}|\widehat{\phi}^{\top}H_0\widehat{\phi}| \le \frac{1}{2}\Lambda \|\widehat{\phi}\|_2^2 \le \Lambda \alpha^2 C_2^2 \tag{134}$$

Secondly, we bound the expectation of J_2 .

We define an event \mathcal{D}_k as follows,

$$\mathcal{D}_k = \bigcap_{j=0}^k \Big\{ \|\theta_j - \widehat{\theta}_j\|_2 \le \alpha^2 \sqrt{\log \frac{1}{\alpha}} \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} C_4, \text{ and } \|\widehat{\theta}_j - \widehat{\theta}_0\|_2 \le \alpha C_2 \Big\},$$

from the result of Lemma 10 and Lemma 12, for each $k \in [0, \hat{\kappa}_0]$, we have

$$\mathbb{P}(\mathcal{D}_k) \ge 1 - \min\left\{\delta, \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}}\right\}.$$

Recall $J_2 = \nabla J(\theta_0)^\top \phi + \hat{\phi}^\top H_0 \phi + \frac{1}{2} \phi^\top H_0 \phi - \frac{\chi}{6} \|\hat{\phi} + \phi\|_2^3$, we need to bound all the terms of J_2 with high probability, In fact,

$$\left|\nabla J(\theta_0)^{\top}\phi\right| \le \epsilon \left\|\theta_T - \widehat{\theta}_T\right\|_2 \stackrel{(126)}{\le} \alpha^2 \sqrt{\log\frac{1}{\alpha}} \sqrt{\frac{\epsilon}{\chi}} \sigma_{H_0} C_4 + o\left(\sqrt{\frac{\epsilon}{\chi}} \alpha^2 \sqrt{\log\frac{1}{\alpha}}\right) \tag{135}$$

$$\begin{aligned} \left| \widehat{\phi}^{\top} H_{0} \phi \right| &\leq \left\| \widehat{\phi} \right\|_{2} \left\| H_{0} \right\|_{op} \left\| \phi \right\|_{2} \stackrel{(91),(126)}{\leq} \Lambda \alpha \sqrt{\alpha^{3} \log \frac{1}{\alpha}} \left[\frac{\log \left(\frac{1}{1 - \sqrt{\alpha} \sigma_{H_{0}}} \right)}{\log (1 + \alpha \sqrt{\chi \epsilon})} \right] C_{4} \alpha C_{2} \\ &= \alpha^{3} \sqrt{\log \frac{1}{\alpha}} \frac{\sigma_{H_{0}}}{\sqrt{\chi \epsilon}} \Lambda C_{2} C_{4} + o \left(\alpha^{3} \sqrt{\log \frac{1}{\alpha}} \cdot \frac{1}{\sqrt{\chi \epsilon}} \right), \quad (136) \end{aligned}$$

$$\left|\frac{1}{2}\phi^{\top}H_{0}\phi\right| \leq \frac{1}{2} \left\|\phi\right\|_{2} \left\|H_{0}\right\|_{op} \left\|\phi\right\|_{2} \stackrel{(126)}{\leq} \frac{1}{2\chi\epsilon} \Lambda^{2}\alpha^{4}\log\frac{1}{\alpha}\sigma_{H_{0}}^{2}C_{4}^{2} + o\left(\frac{1}{\chi\epsilon}\alpha^{4}\log\frac{1}{\alpha}\right)$$
(137)

$$\frac{\chi}{6} \|\widehat{\phi} + \phi\|_2^3 \le \frac{\chi}{6} \left(\sqrt{\alpha^3 \log \frac{1}{\alpha}} \left\lfloor \frac{\log \left(\frac{1}{1 - \sqrt{\alpha} \sigma_{H_0}}\right)}{\log(1 + \alpha \sqrt{\chi \epsilon})} \right\rfloor C_4 + \alpha C_2 \right)^3 = \frac{\chi}{6} \alpha^3 C_4^3 + o(\alpha^3).$$
(138)

From the results of (135)-(138), for a enough small α , we have

$$J_2 \mathbf{1}_{\mathcal{D}_k} \le \alpha^2 \sqrt{\log \frac{1}{\alpha}} \sqrt{\frac{\epsilon}{\chi}} \sigma_{H_0} C_4 + o\left(\sqrt{\frac{\epsilon}{\chi}} \alpha^2 \sqrt{\log \frac{1}{\alpha}}\right),$$

which implies

$$\mathbb{E}[J_2 \mathbf{1}_{\mathcal{D}_k}] \le \alpha^2 \sqrt{\log \frac{1}{\alpha}} \sqrt{\frac{\epsilon}{\chi}} \sigma_{H_0} C_4 + o\left(\sqrt{\frac{\epsilon}{\chi}} \alpha^2 \sqrt{\log \frac{1}{\alpha}}\right).$$
(139)

Furthermore, from the result of (65), (69), it is easy to show $\|\theta_k - \hat{\theta}_k\|_2 \leq \mathcal{O}(1)$, $\|\theta_k - \hat{\theta}_0\|_2 \leq \mathcal{O}(1)$, which implies

$$\mathbb{E}[J_2 \mathbf{1}_{\overline{\mathcal{D}_k}}] \le \min\left\{\delta, \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}}\alpha^{\frac{3}{2}}\right\}.$$
(140)

By the results of (139) and (140), the lower boundedness of $\mathbb{E}[J_2]$ reaches

$$\mathbb{E}[J_2] \ge -\alpha^2 \sqrt{\log \frac{1}{\alpha}} \sqrt{\frac{\epsilon}{\chi}} \sigma_{H_0} C_4 - \min\left\{\delta, \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}}\right\} = \mathcal{O}\left(-\min\left\{\delta, \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}}\right\}\right).$$

Combining above result with (133), we have

$$\mathbb{E}[J(\theta_T)] - J(\theta_0) \ge \mathbb{E}[J_1] + \mathbb{E}[J_2]$$

$$\ge \alpha (1 + \alpha \sqrt{\chi \epsilon}) \iota^2 - \Lambda \alpha^2 C_2^2 - \mathcal{O}\Big(\min\Big\{\delta, \frac{\sigma_{H_0}}{\sqrt{\chi \epsilon}} \alpha^{\frac{3}{2}}\Big\}\Big)$$

$$\ge \alpha^2 \iota^2 \sqrt{\chi \epsilon}, \tag{141}$$

the last (141) holds since we chose a proper step-size satifies

$$\alpha \iota^2 - \Lambda \alpha^2 C_2^2 - \mathcal{O}\left(\min\left\{\delta, \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}}\right\}\right) > 0, \tag{142}$$

for a small α , Eq.(142) always exiits. Concretely, if $\delta > \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}}$, then we can chose step as follows,

$$\sqrt{\alpha} < \frac{\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} + \sqrt{(\frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}})^2 + 4\Lambda C_2^2 \iota^2}}{a\Lambda C_2^2};$$
(143)

otherwise, if $\delta \leq \frac{\sigma_{H_0}}{\sqrt{\chi\epsilon}} \alpha^{\frac{3}{2}}$, we can chose step as follows,

$$\alpha \iota^2 - \Lambda \alpha^2 C_2^2 - \delta \ge \alpha \iota^2 - \Lambda \alpha^2 C_2^2 - \frac{\sigma_{H_0}}{\sqrt{\chi \epsilon}} \alpha^{\frac{3}{2}} > 0,$$

which implies same condition as Eq.(143).

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